Calculus for Everyone (Chapters 1 - 11)

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Contents

Preface

> As its campfires glow against the dark, every culture tells stories to itself about how the gods lit up the morning sky and set the wheel of being into motion. The great scientific culture of the west - our culture - is no exception. The calculus is the story this world first told itself as it became the modern world. - David Berlinski

> But how is one to make a scientist understand that there is something unalterably deranged about differential calculus, quantum theory, or the obscene and so inanely liturgical ordeals of the precession of the equinoxes. - Antonin Artaud (said tongue-in-cheek no doubt)

Regardless of your Calculus pursuasion, like Shakespeare's plays everyone should see Calculus at least once in their schooling. Caculus is the language of movement. Anyone who wishes to learn Calculus can learn it. However, not everyone can read a Calculus textbook. This is because most math textbooks are written for the instructor, not the student. They are comprehensive tomes, often over 500 pages, and written so that an instructor can make a selection of material for a semester based on personal taste. This book is for the student taking Calculus. The selection of material is already made. All the material presented is essential (besides the nerdy jokes and cultural asides).

The advantage of having a book like this as the textbook is that the instructor can expect the students to actually read the material and come prepared to class having done the "readings." Every effort is made to make the book readable. The standard textbook format where Chapter 1 is divided into sections, and sections into subsections and so on, is eschewed in favor of a more narrative style. The standard format with examples labeled 1.3.1.2 is good for an instructor who uses the textbook as a reference manual, but to the student it reads like a dry code manual. In this book there are no sections and subsections, just chapters like in a novel.

The graphs in this book are produced using wolframalpha.com which is the free version of Mathematica. This website has a Google like interface. An equation or function can be entered into the box somewhat imprecisely and it will understand and solve it or sketch it. This is a bigger advantage than it sounds. Having to know precise codes for everything makes using graphic calculators and computer algebra systems tedious. Even one bracket out of place causes an error. Free software that is intelligent enough to understand what the user intends to type is a game-changer in the way students can learn Precalculus and Calculus.

Figures in the book have a simple appearance to emphasize that they can be drawn by hand. Modern textbook have an impressive array of pictures and figures that are not only attention-deficit inducing, but also leave the reader with the sense that he could never draw the figures himself. Conventional wisdom supposes that some people understand information better when it is presented with words, whereas others prefer a more visual approach. But in mathematics, the visual approach is not just an alternate pedagogical approach, it is essential. We understand mathematical objects in terms of pictures. For example, the object $y = x^2$ is not really meaningful unless it is recognized as the parabola with vertex $(0, 0)$ that is symmetric about the y-axis. Arguably we use words to convey the pictures we have in mind.

Right from the beginning students are encouraged to find solutions to problems on their own using freely available resources. The exercises do not come with a solution sheet. In most cases students can use wolframalpha.com to get the solutions. Students should learn to work with friends to check their solution and with practice become confident that their work is correct. This book works well for teaching strategies such as "collaborative learning" and "team-based learning," which emphasize working in groups. It also works well for the "flipped classroom," where the students have to read and understand each chapter independently before it is presented in the classroom and class-time is spent discussing the material not teaching it. It may not be a good idea to completely forgo the old-fashioned lecture style of teaching mathematics in favor of a new strategy. However, including active learning components like allowing students to ask questions, requiring them to read ahead and come prepared to discuss the material, and giving team-based quizzes are effective teaching and learning tools.

This book has few word problems. Word problems in math textbooks are proxy for real-world applications. They are contrived and often confusing paragraphs that supposedly capture a situation where math is applied to a real-world situation, but they manage to remove all the fun out of both the math and the application. The difficulty with doing applications properly in a math class is that the applications are discipline specific and reqire considerable subject knowledge. There is no time in the mathematics classroom to give students a solid application from another discipline, without sacrficing something, and that something is usually rigor. The best course of action for students is to first work through a few carefully worded problems like the ones in this book to get the ideas straight. As Hardy said in his 1941 book "A Mathematician's Apology,"

> A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

Then, based on interest, follow up by looking at one of the websites that specialize in applications that use Calculus. Don't waste time on artifical real-word problems (the oxymoron is unavoidable). Theory and applications must go hand-in-hand despite an educational system seemingly set up to make this impossible.

Typically Chapters 1 to 12 (chapter 12 is still unwritten) covering limits and derivatives corresponds to a 3 or 4 credit Calculus I course lasting 15 weeks, and integration and series is covered in a Calculus II course. There is considerable scaffolding of topics to gradually move students toward a stronger understanding, ultimately making them independent learners. This is why the proofs of theorems appear in a separate chapter at the end. These proofs are essential for Calculus II, but not essential for a student who just wants a taste of Caculus. The proofs become more meaningful after they are used in problem solving. So the focus throughout is solving problems.

Many students require only a Calculus I course. Putting students who require only Calculus I in the same class as those who require Calculus I and II causes major problems. If limits and derivatives are covered rigorously, then students never see the beauty and importance of integration. If a non-rigorous approach is adopted, then the students going on to Calculus II have too weak a background to succeed. It is better to offer students who require only one calculus course a "Survey of Calculus" that covers both derviatives and integrals and is known from the beginning to be a "mile-wide and inch-deep." This book can be used for such a course by studying limits without emphasizing techniques for finding them, omitting all proofs as well as the more complicated derivatives and integrals, and completely omitting series. Throughout there are footnotes to help students navigate the material in a non-rigourous fashion. The hope is that such students become interested in knowing things thoroughly when they realize they can read a math textbook on their own.

We will end with Steven Strogatz's description of Calculus taken from his wonderful New York Times article "Change we can believe in" that should be required reading for everyone.

> Calculus is the mathematics of change. It describes everything from the spread of epidemics to the zigs and zags of a well-thrown curveball. The subject is gargantuan - and so are its textbooks. Many exceed 1,000 pages and work nicely as doorstops. But within that bulk you'll find two ideas shining through. All the rest, as Rabbi Hillel said of the Golden Rule, is just commentary. Those two ideas are the "derivative" and the "integral." Each dominates its own half of the subject, named in their honor as differential and integral calculus. Roughly speaking, the derivative tells you how fast something is changing; the integral tells you how much it's accumulating. They were born in separate times and places: integrals, in Greece around 250 B.C.; derivatives, in England and Germany in the mid-1600s. Yet in a twist straight out of a Dickens novel, they've turned out to be blood relatives - though it took almost two millennia to see the family resemblance.

The author thanks Professor Miriam Deutch for taking the lead on OpenSource@CUNY, the faculty and technical staff on the team, and Brooklyn College for giving a 3-credit course release to write yet another commentary on one of the greatest discoveries of humanity.

CHAPTER 1

What is a Limit?

Calculus is the study of change. It is a way of making sense of the world we live in. If this utilitarian argument doesn't appeal to you, then learn Calculus because it is one of the great achievements of humanity. Study it like you would a great piece of artwork, except that it isn't tangible and gets drawn in your head with the help of ideas. It does, however, have considerable prerequistes including algebra, geometry, and trigonometry grouped together under the umbrella term "Precalculus." So, assuming you have taken a Precalculus course let's begin, as the King of Hearts from *Alice in Wonderland* said, "at the beginning." which really isn't the begining at all as we will see later on, but we have to start somewhere.

EXAMPLE 1.1. Consider the function $f(x) = (x-2)^2$ whose graph is shown in Figure 1.0.1.

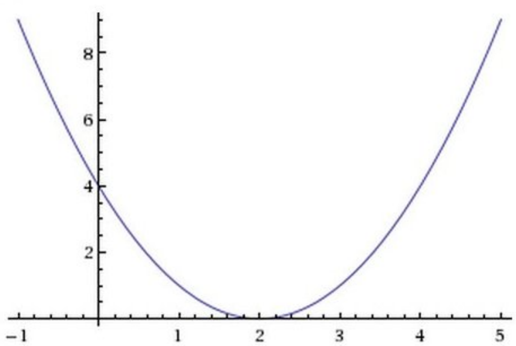


FIGURE 1.0.1. $f(x) = (x-2)^2$

Observe from the graph [1] that

- As x tends to 2 from the left, $f(x)$ tends to 0;
- As x tends to 2 from the right, $f(x)$ tends to 0.

Graphs in Calculus are "moving images." Obviously the graph in Figure 1.0.1 is static, but we must imagine a point on the x-axis moving toward 2 from the left, and at the same

[1] You should be able to draw the graph of this function by hand. Otherwise, go to wolframalpha.com and enter the function into the textbox as `f(x) = (x-2) caret symbol 2`. Mathematica will draw a graph with a domain and range it considers suitable. The precise Mathematics code for graphing is `Plot[f(x), {x, min, max}, {y, min, max}]`.

time a point on the graph of $f(x)$ moving along the curve toward 0. We use the symbol 2^- to denote the phrase "2 from the left" and 2^+ to denote the phrase "2 from the right." We use $\to$ for the phrase "tends to." The above observation in symbols becomes

- As $x \longrightarrow 2^-$, $f(x) \longrightarrow 0$;
- As $x \longrightarrow 2^+$, $f(x) \longrightarrow 0$.

We further shorten it using the "limit" notation as follows:

- $\lim_{x\to 2^-} f(x) = 0$ (read as "limit as x tends to 2 minus of f of x is 0");
- $\lim_{x\to 2^+} f(x) = 0$.

We call $\lim_{x\to 2^-} f(x) = 0$ the **left limit** and $\lim_{x\to 2^+} f(x) = 0$ the **right limit**. Observe that the left limit and right limit both exist and are equal. So

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 0.$$

In this case we say the **limit exists** and we write

$$\lim_{x\to 2} f(x) = 0.$$

Example 1.2. Consider the piecewise function $g(x) = \begin{cases} 3x & \text{if } x \neq 2 \\ 8 & \text{if } x = 2 \end{cases}$ whose graph is shown in Figure 1.0.2.

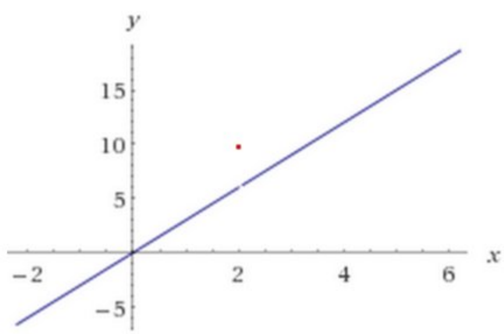


Figure 1.0.2. The piecewise function $g(x)$ with a hole at 2

A piecewise function is defined in different pieces on the domain; hence the name. This function $g(x)$ is the line $y = 3x$ with a "hole" in it. The hole exists because $3(2) = 6$ but at $x = 2$ the function takes the value 8. [2]

[2] Piecewise functions have to be entered precisely using Mathematica Code. Enter $g(x)$ as `g(x) = Piecewise[{{3x, x[NotEqual]2}, {8, x=2}}]`.

Observe that

$$\lim_{x \to 2^-} g(x) = 6 \text{ and } \lim_{x \to 2^+} g(x) = 6.$$

So,

$$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^+} g(x) = 6.$$

We say the **limit** exists and we write

$$\lim_{x \to 2} g(x) = 6.$$

Observe that, in the first example $\lim_{x \to 2} f(x) = f(2)$ and in the second example $\lim_{x \to 2} g(x) \neq g(2)$ because $\lim_{x \to 2} g(x) = 6$ and $g(2) = 8$. However, in both cases the limit exists.

Example 1.3. Consider the piecewise function $h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$ whose graph is shown in Figure 1.0.3.

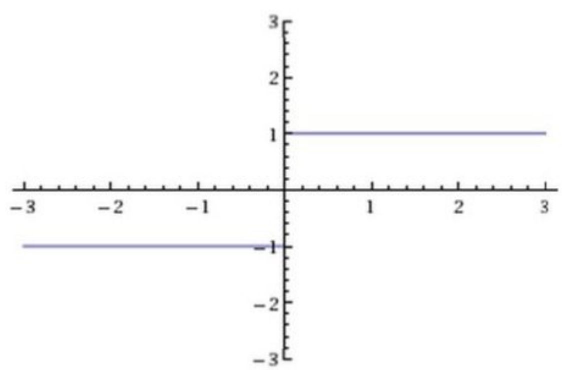


Figure 1.0.3. The step function $h(x)$

This piecewise function is defined one way on the interval $[0, \infty)$ and another way on $(-\infty, 0)$. It is called a **step function** because it looks like a step. [3] Observe that

$$\lim_{x \to 0^-} h(x) = -1 \text{ and } \lim_{x \to 0^+} h(x) = 1.$$

In this example

$$\lim_{x \to 0^-} h(x) \neq \lim_{x \to 0^+} h(x)$$

When the left and right limits are different, we say $\lim_{x \to 0} h(x)$ **does not exist** or in short "**dne.**" Look carefully at $h(x)$ as it is our first example where the limit does not exist.

[3]The Mathematica Code for entering this function is

```
h(x) = Piecewise[{{1, x >= 0}, {-1, x < 0}}] x from -3 to 3 y from -3 to 3
```
.

EXAMPLE 1.4. Consider the rational function $k(x) = \frac{1}{x}$ whose graph is shown in Figure 1.0.4.

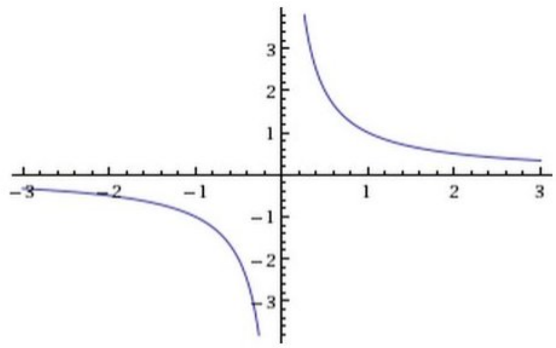


FIGURE 1.0.4. The hyperbola $k(x) = \frac{1}{x}$

This function is called a hyperbola.[4] The x-axis is the **horizontal asymptote** and the y-axis is the **vertical asymptote**. It is not defined at 0 because otherwise we would have 0 in the denominator which is not permissible. [5] Observe that

$$\lim_{x\to 0^-} k(x) = -\infty \text{ and } \lim_{x\to 0^+} k(x) = \infty.$$

Since

$$\lim_{x\to 0^-} k(x) \neq \lim_{x\to 0^+} k(x)$$

we may conclude that

$$\lim_{x\to 0} k(x) \text{ does not exist.}$$

Observe further that

$$\lim_{x\to \infty} k(x) = 0 \text{ and } \lim_{x\to -\infty} k(x) = 0.$$

We define a horizontal asymptote in the above manner.

Definition of Horizontal Asymptote

Let f be a function and $y = k$ be a horizontal line, where k is some real number. We say $y = k$ is a **horizontal asymptote** for f, if $\lim_{x\to\infty} f(x) = k$ or $\lim_{x\to-\infty} f(x) = k$.

EXAMPLE 1.5. Consider the rational function $l(x) = \frac{1}{x^2}$ whose graph is shown in Figure 1.0.5.

[4]Enter $k(x)$ as `k(x) = 1/x`.

[5]As Steven Wright said "Black holes are where God divided by zero."

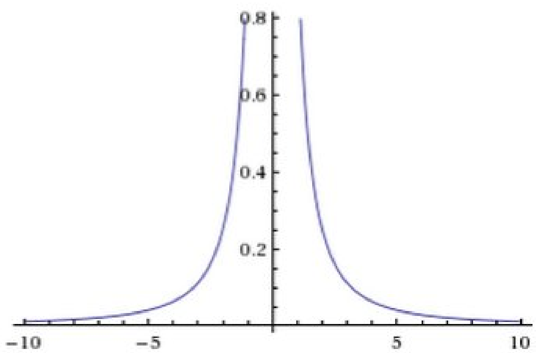


FIGURE 1.0.5. The hyperbola $l(x) = \frac{1}{x^2}$

Observe that

$$\lim_{x \to 0^-} l(x) = \infty \text{ and } \lim_{x \to 0^+} l(x) = \infty.$$

In this example, since the left and right limits exist and are the same,

$$\lim_{x \to 0} l(x) = \infty$$

Note that we are abusing notation slightly when we say the limit is equal to ∞ since infinity is not a number. Some textbooks prefer to use words saying "the limit tends to infinity." Other textbooks prefer to be brief and (mis)use the ∞ notation. We will adopt the latter convention.

We are now ready to formulate a working definition of limits.

Definition of Limit

Let $f(x)$ be a real-valued function defined on an open interval containing point a, except possibly at a, and let L be a real number.

(1) If $f(x)$ tends to L as x tends to a from the left, then the **left limit** exists and we write $\lim_{x \to a^-} f(x) = L$

(2) If $f(x)$ tends to L as x tends to a from the right, then the **right limit** exists and we write $\lim_{x \to a^+} f(x) = L$

(3) If the left and right limit both exists and are equal, then the **limit** exists and we write $\lim_{x \to a} f(x) = L$

(4) If $f(x)$ tends to L as x tends to ∞, then $\lim_{x \to \infty} f(x) = L$

This definition is what we call an **informal definition**. It will do for now, but soon we will see why it is inadequate. Let us see some more examples.

Example 1.6. Find the following limits:

(1) $\lim_{x \to 2} 5$

(2) $\lim_{x \to 2} 3x + 7$

(3) $\lim_{x \to -\infty} 2^x$

(4) $\lim_{x \to 0} ln(x)$

(5) $\lim_{x \to \frac{\pi}{2}} \tan x$

(6) $\lim_{x \to 0} \cot x$

(7) $\lim_{x \to 0} \csc x$

(8) $\lim_{x \to \infty} \tan^{-1} x$

Solution.

(1) The graph of $y = 5$ is a horizontal straight line (see Figure 1.0.6). No matter what value x takes, whether it is 2 or something else, y is always 5. So $\lim_{x \to 2} 5 = 5$

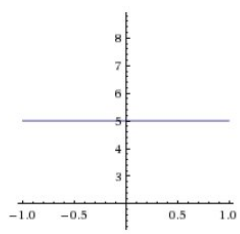


Figure 1.0.6. The horizontal line $y = 5$

(2) The graph of $y = 3x + 7$ is the straight line shown in Figure 1.0.7. Observe that $\lim_{x \to 2} 3x + 7 = 3(2) + 7 = 13$.

(3) The graph of $y = 2^x$ is shown in Figure 1.0.8. Observe that the x-axis is the horizontal asymptote. So $\lim_{x \to -\infty} 2^x = 0$.

(4) The graph of $y = \ln x$ is shown in Figure 1.0.9. Observe that the y-axis is the vertical asymptote. So $\lim_{x \to 0} \ln x = -\infty$.

(5) The graph of $y = \tan x$ is shown in Figure 1.0.10. It has a vertical asymptote at $x = \frac{\pi}{2}$. Observe that $\lim_{x \to \frac{\pi}{2}^-} \tan x = -\infty$ and $\lim_{x \to \frac{\pi}{2}^+} \tan x = \infty$. So $\lim_{x \to \frac{\pi}{2}} \tan^{-1} x$ does not exist.

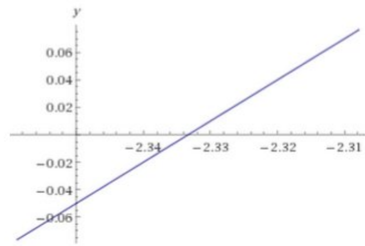


FIGURE 1.0.7. The line $y = 3x + 5$

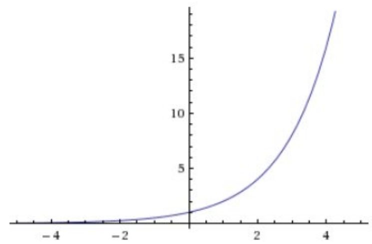


FIGURE 1.0.8. $y = 2^x$

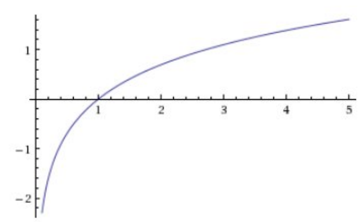


FIGURE 1.0.9. $y = \ln x$

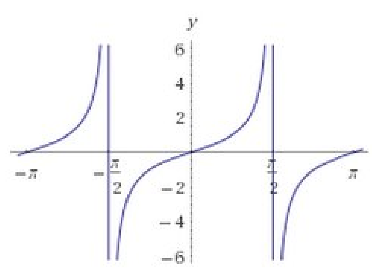


FIGURE 1.0.10. $y = \tan x$

(6) The graph of $y = \cot x$ is shown in Figure 1.0.11. It has a vertical asymptote at $x = 0$. Observe that $\lim_{x \to 0} \cot x$ does not exist

(7) The graph of $y = \csc x$ is shown in Figure 1.0.12. It has a vertical asymptote at $x = 0$. Observe that $\lim_{x \to 0} \csc x$ does not exist.

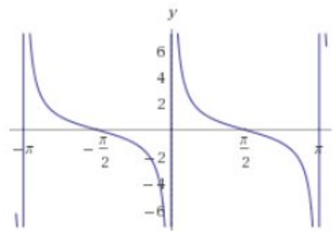


FIGURE 1.0.11. $y = \cot x$

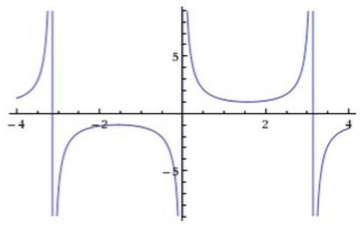


FIGURE 1.0.12. $y = \csc x$

(8) The graph of $y = \tan^{-1} x$ is shown in Figure 1.0.13. It has a horizontal asymptote at $y = \frac{\pi}{2}$. Observe that $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$. $\square$

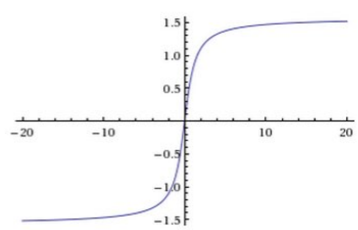


FIGURE 1.0.13. $y = \tan^{-1} x$

Note that in the last example just looking at the graph of $f(x) = \tan^{-1} x$ with no understanding of the inverse tangent function (also called arctan) will not lead to the answer. The function $f(x) = \tan^{-1} x$ is not one-to-one (see Figure 1.0.10). But if we restrict it to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, then the restriced tangent function is one-to-one. It has domain $(-\frac{\pi}{2}, \frac{\pi}{2})$, range $(-\infty, \infty)$, and vertical asymptotes $x = \pm\frac{\pi}{2}$. The inverse of this restricted tangent function is obtained by reflecting it along the line $y = x$. It is precisely the function shown in Figure 1.0.13 with domain $(-\infty, \infty)$, range $(-\frac{\pi}{2}, \frac{\pi}{2})$, and horizontal asymptotes $y = \pm\frac{\pi}{2}$. By definition of horizontal asymptote $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$. This is not a formal proof of the limit, but a better explanation than guessing from the graph. [6]

[6]Limits are infused into the popular cultural making appearances on TV shows and films. There is a scene in Big Bang Theory, Episode 213 "The Friendship Algorithm," that refers to $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$.

Wolowitz: Gee, why can't Sheldon get a friend?"

x	$y = \frac{sinx}{x}$
± 0.1	0.998334166
± 0.01	0.99998333
± 0.001	0.999998333
± 0.0001	0.9999995
± 0.00001	1.0
± 0.000001	1.0

TABLE 1. The behavior of $f(x) = \frac{sinx}{x}$ near $x = 0$

So far we adopted a highly visual approach to limits. But drawing graphs is not always possible.

EXAMPLE 1.7. Consider the function $f(x) = \frac{sinx}{x}$ whose graph is shown in Figure 1.0.14 (but don't look now).

This is not a function with a well-known shape like the previous functions. How should we find $\lim\limits_{x\to 0} \frac{sinx}{x}$? One approach is to take values of x closer and closer to 1 and find the corresponding values of $\frac{sinx}{x}$. This is the computational approach and it is illustrated in Table 1. Observe that as x tends to 0, $\frac{sinx}{x}$ tends to 1. The computational approach *suggests* that

$$\lim_{x\to 0} \frac{sinx}{x} = 1.$$

But, how do we know something strange doesn't happen very close to 0. There is clearly something stronger than computation needed to say conclusively that the limit is 1.

As mentioned earlier, the graph of $f(x) = \frac{\sin x}{x}$ is shown in Figure 1.0.14. The graph supports our computational intuition that the limit is 1. Are we correct? The answer is yes in this case, and it is quite clear from the graph. However, we are left with a vague sense that more is needed.

The previous example may suggest that using graphing software will always help in finding limits. Unfortunately, this is not the case.

EXAMPLE 1.8. Consider the function $f(x) = \sin \frac{\pi}{x}$ whose graph is shown in Figure 1.0.15.

Judging by this graph, we might hesitantly venture a guess that $\lim\limits_{x\to 0} \sin \frac{\pi}{x}$ is 0. Table 2 shows that our guess is bolstered by computational evidence. If we tried to zoom in to get a clearer picture of the function's behavior near zero we would get Figure 1.0.16, which is no more helpful. So are we correct? The answer is no.

$$\lim_{x\to 0} \sin \frac{\pi}{x} \text{ does not exist.}$$

Sheldon: What part of an inverse tangent approaching an asymptote don't you understand?

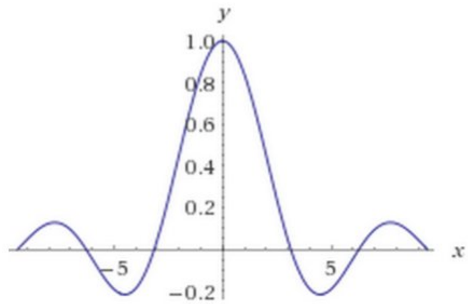


FIGURE 1.0.14. $f(x) = \frac{\sin x}{x}$

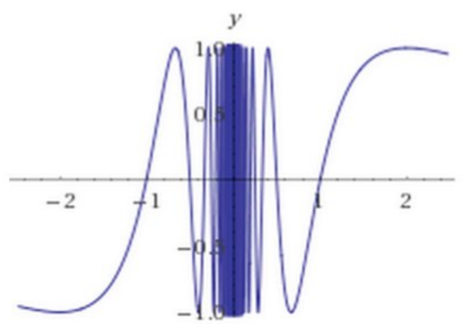


FIGURE 1.0.15. $f(x) = \sin \frac{\pi}{x}$

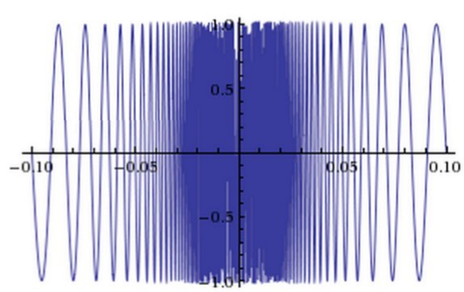


FIGURE 1.0.16. A zoomed in view of $f(x) = \sin \frac{\pi}{x}$ near 0

This is because no matter how close x gets to 0 the function $f(x)$ oscillates *wildly* from -1 to 1. This is not based on visual intuition nor computational intuition. A stronger definition is needed and that is the formal definition of the limit.

x	$y = \sin \frac{\pi}{x}$
1	0
$\frac{1}{2}$	0
$\frac{1}{3}$	0
$\frac{1}{4}$	0
$\frac{1}{5}$	0
$\frac{1}{6}$	0

TABLE 2. The behavior of $f(x) = \sin \frac{\pi}{x}$ near $x = 0$

Formal Definition of Limit

Let $f(x)$ be a real-valued function defined on an open interval containing a point a, except possibly at a, and let L be a real number. The statement

$$\lim_{x \to a} f(x) = L$$

means that for every $\epsilon > 0$, there exists $\delta > 0$, such that

$$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon.$$

Figure 1.0.17 [7] explains the definition. For every small number ϵ such that the distance of $f(x)$ to L is less than ϵ, there is a small number δ such that the distance between x and a is less than δ. Note that $|x - a| < \delta$ means x is in the interval $(a - \delta, a + \delta)$ and $|f(x) - L| < \epsilon$ means $f(x)$ is in the interval $(L - \epsilon, L + \epsilon)$. As the interval around L on the y-axis shrinks, the interval around a on the x-axis shrinks. The box in the center of the figure also shrinks and captures the limit inside it.

The formal definition of limits is mostly outside the scope of this course, but it helps to have some understanding of it (you will begin an Advanced Calculus with this definition). As usual the way to understand something is to work out some examples. The examples here are “proofs" and we postpone it till Chapter 12. At this junction it is enough to realize that something stronger than the visual and computation approach to limits is needed.

[7] This figure is taken from `https://www.math.ucdavis.edu/ kouba/CalcOneDIRECTORY`, which also has several examples of how the formal definition of limits is used to find limits and prove that the limit does not exist.

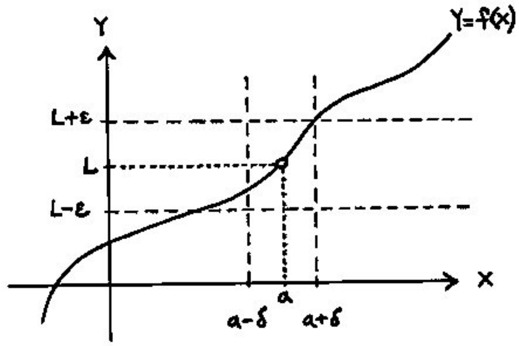


FIGURE 1.0.17. The $\epsilon - \delta$ definition of limits

We end this chapter by looking at another famous example of limits in the media. This limit appears in the 2004 movie "Mean Girls" directed by Tina Fey. [8] Find

$$\lim_{x\to 0} \frac{\ln(1-x) - \sin x}{1-\cos^2 x}.$$

There is a rigorous way of finding this limit, which we will see later, but assume for the moment that we only have the visual technique. How could we figure out the limit? The graph of this function isn't one that anyone remembers, so graphing isn't an option. It would make sense to quickly plug 0 in the function to see what happens. But we get $\frac{0}{0}$ which is not helpful (at this stage). However, a quick simplication of the function (can be done mentally) gives us

[8] The story line in brief: Cady Heron lived in Africa and is homeschooled until she comes to the United States and goes to North Shore High School. She is a nice girl who befriends the popular mean girls, and subsequently becomes meaner than the meanest girl. Since she is good at math she gets invited to join the Mathletes team; albeit in a disturbing way that reveals social norms in typical tina Fey fashion.

> Student: I'm Kevin Gnapoor, Captain of the North Shore Mathletes. We participate in math challenges against other high schools in the state, and we can get twice as much funding if we've got a girl. So you should think about joining.

At the end of the movie, the Matheletes participate in the state competition. The teams are asked to find the above limit. The opposing team rings the bell first, but answers incorrectly. Cady answers the question correctly making the NorthShore Mathletes the state champions. As she does it she also realizes the foolishness of her mean ways. The MovieMath website has useful information on math in the movies. See `http://www.qedcat.com/moviemath/mean(underscore)girls.html`. The clip may be watched on YouTube `http://kasmana.people.cofc.edu/MATHFICT/mfview.php?callnumber=mf450`.

$$\begin{aligned}
\lim_{x\to 0}\frac{\ln(1-x)-\sin x}{1-\cos^2 x} &= \lim_{x\to 0}\frac{\ln(1-x)-\sin x}{\sin^2 x}\\
&= \lim_{x\to 0}\frac{\ln(1-x)}{\sin^2 x}-\frac{\sin x}{\sin^2 x}\\
&= \lim_{x\to 0}\frac{\ln(1-x)}{\sin^2 x}-\frac{1}{\sin x}\\
&= \lim_{x\to 0}\frac{\ln(1-x)}{\sin^2 x}-\csc x
\end{aligned}$$

We should know the graph of $y = \csc x$ from memory (see Figure 1.0.13) and by looking at the graph we can conclude

$$\lim_{x\to 0}\csc x \text{ does not exist.}$$

Therefore it is reasonable (if not precisely accurate) to conclude that

$$\lim_{x\to 0}\frac{\ln(1-x)-\sin x}{1-\cos^2 x} \text{ does not exist.}$$

It is easy to see from the graph that the limit does not exist. See Figure 1.0.18. [9]

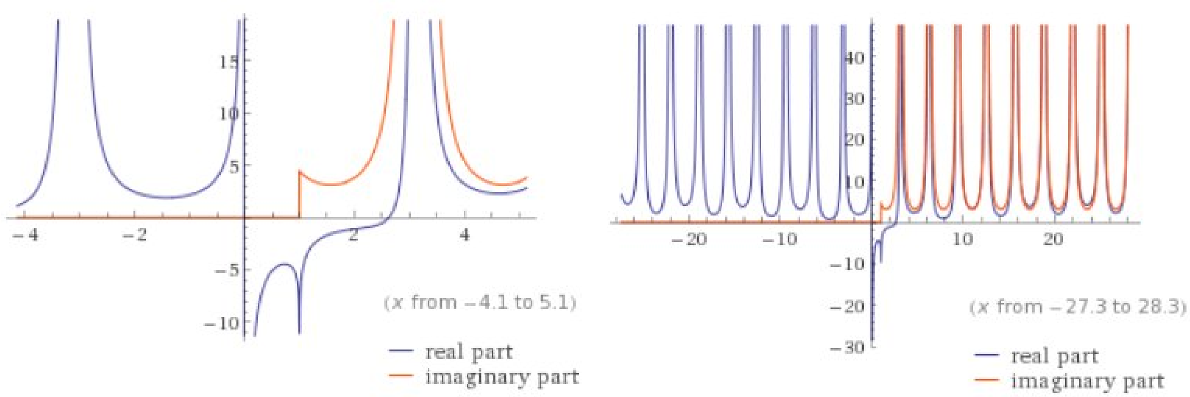


FIGURE 1.0.18. The limit problem in the competition

Practice Problems

(1) Draw the graph of the function using your "by-hands" graphing techniques and find the following limits. If the limit does not exist, explain why using your graph.

[9] Enter this function as `Plot [(ln(1-x)-sinx)/(1-cos(caretsymbol)2(x))]`. This function has its real and imaginary parts closely intertwined; hence the blue and red curves.

a) $\lim_{x\to 2} 3x+1$
b) $\lim_{x\to 10} \frac{1}{x}$
c) $\lim_{x\to \infty} \frac{1}{x}$
d) $\lim_{x\to -\infty} \frac{1}{x}$
e) $\lim_{x\to 0} \frac{1}{x}$
f) $\lim_{x\to 0} \frac{1}{x^3}$
g) $\lim_{x\to 2} \frac{1}{x-2}$
h) $\lim_{x\to \infty} \frac{1}{x-2}$
i) $\lim_{x\to 0} \frac{1}{x}+5$
j) $\lim_{x\to \infty} \frac{1}{x}+5$
k) $\lim_{x\to \infty} \frac{1}{x}-7$
l) $\lim_{x\to 0} \sin x$
m) $\lim_{x\to 0} \cos x$
n) $\lim_{x\to \frac{\pi}{2}} \tan x$
o) $\lim_{x\to 2} \frac{|x|}{x}$
p) $\lim_{x\to 0} \frac{|x|}{x}$
q) $\lim_{x\to 0^+} \sqrt{x}$
r) $\lim_{x\to 0^+} \log_2 x$

(2) Draw the graphs using software and find the following limits. If the limit does not exist, explain why using your graph.

a) $\lim_{x\to 0} \frac{\sin x}{x}$
b) $\lim_{x\to 0} \frac{\cos x}{x}$
c) $\lim_{x\to 0} \sin \frac{1}{x}$

(3) The following functions are piece-wise functions. Draw the graphs using "by-hands" graphing techniques and find the following limits.

a) $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ where:

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

b) $\lim_{x\to 1} f(x)$ where:

$$f(x) = \begin{cases} 3-x & x < 1 \\ 4 & x = 1 \\ x^2+1 & x > 1 \end{cases}$$

c) $\lim_{x\to n} f(x)$, where n is an integer and

$$f(x) = \begin{cases} x & x = n \\ 0 & x \neq n \end{cases}$$

d) $\lim_{x\to n^-} f(x)$ and $\lim_{x\to n^+} f(x)$, where n is an integer and $f(x) = [x]$.

e) $\lim_{x\to n^-} f(x)$ and $\lim_{x\to n^+} f(x)$, where n is an integer such that $n \leq x \leq n+1$ and $f(x) = (-1)^n$.

CHAPTER 2

Continuous Functions

> The continuous function is the only workable and usable function. It alone is subject to law and the laws of calculation. It is a loyal subject of the mathematical kingdom. Other so-called or miscalled functions are freaks, anarchists, disturbers of the peace, malformed curiosities which one and all are of no use to anyone, least of all to the loyal and burden-bearing subjects who by keeping the laws maintain the kingdom and make its advance possible. - E. D. Roe, Jr. (A generalized definition of limit, Math. Teacher 3(1910) p.47)

In this chapter we will introduce the concept of continuous functions - the nice and well-behaved functions. Returning to the five intial examples in Chapter 1, in the first example, the left and right limits exist and are both equal to 0 so

$$\lim_{x\to 2} f(x) = 0 = f(2).$$

Note that zero is precisely the value of $f(2)$. In the second example, again the left and right limits exist and are equal to 6, but $g(2) = 8$. Here the limit is not equal to the function value at 2. So

$$\lim_{x\to 2} g(x) = 6 \neq g(2).$$

In the third and fourth example, the left and right limits are different, so $\lim_{x\to 0} h(x)$ and $\lim_{x\to 0} k(x)$ do not exist.

The first function $f(x)$ is called a **continuous function**. If we traced the graph of $f(x)$ in Figure 1.0.1 we can cover the entire curve without raising pencil from paper. This is not the case for $g(x)$ because it has a hole in it. The function $h(x)$ has a step at 0 where it jumps from -1 to 1. The pencil must be raised from paper to get from the left side of 0 to the right side. For the function $k(x)$, as we approach 0 from the left, the function goes to $-\infty$, and from the right it goes to ∞. It is not a hole in the function as in the second example, nor is it a jump as in the third example, but there is a gap at 0 and we must lift pencil from paper. The functions $g(x)$, $h(x)$, and $k(x)$ are not continuous or **discontinuous functions**. We will define continuity formally in terms of limits, but the intuitive definition of not having to lift pencil from paper while tracing the function is very useful.

Definition of Continuity

Let f be a real-valued function and a be a real number. We say $f(x)$ is **continuous at** a if $\lim_{x\to a} f(x) = f(a)$. A function that is continuous at every point in its domain is called a **continuous function**.

By the above definition, $f(x) = (x-2)^2$ is continuous and the piecewise functions $g(x)$ and $h(x)$ are clearly discontinuous. There is a subtlety for the rational function $k(x) = \frac{1}{x}$. Clearly there is a gap at 0 with one side of the function going off to $-\infty$ and the other side going off to ∞. So it is correct to say that $k(x)$ is discontinuous over the interval $(-\infty, \infty)$ because of its behavior at 0, but it is also correct to say that $k(x)$ is continuous on its domain since 0 is not in its domain. This is a matter of nomenclature - making clear what is called what. We say $f(x)$ is **left continuous** at a if $\lim_{x\to a^-} f(x) = f(a)$ and **right continuous** at a if $\lim_{x\to a^+} f(x) = f(a)$. If $f(x)$ is defined only on one side of an endpoint of the domain, then we understand continuous at the end point to mean continuous from that side.

EXAMPLE 2.1. Consider the function $f(x) = \sqrt{x}$ whose graph is shown in Figure 2.0.1.

The domain of this function is $[0, \infty)$. [1] Since it is defined only to the right of 0, it doesn't make sense to talk of left continuity at 0. We conclude $f(x)$ is a continuous function.

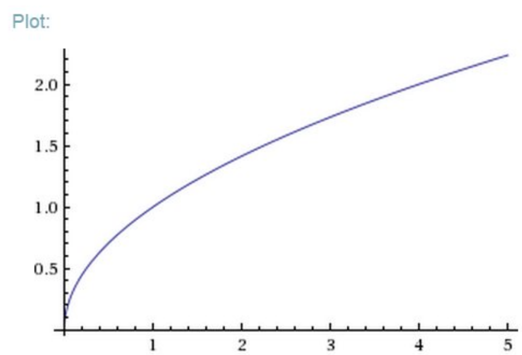


FIGURE 2.0.1. The square root function $f(x) = \sqrt{x}$

There are three types of discontinuity (see Figure 2.0.2).

(1) Removable discontinuity: when the function has a hole at a point t, but also has a limit at t, as in the piecewise function from the second example

$$g(x) = \begin{cases} 3x & \text{if } x \neq 2 \\ 8 & \text{if } x = 2 \end{cases}$$

The function may or may not be defined at t.

(2) Jump discontinuity: when the function has a jump at a point t and the limit does not exist at t, as in the piecewise function from the third example

$$h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

[1] Enter this function in wolframalpha.com as `f(x)=sqrt(x) x from 0 to 5`.

(3) Infinite discontinuity: when the function goes to $-\infty$ on one side and ∞ on the other, as in the rational function from the fourth example

$$k(x) = \frac{1}{x}$$

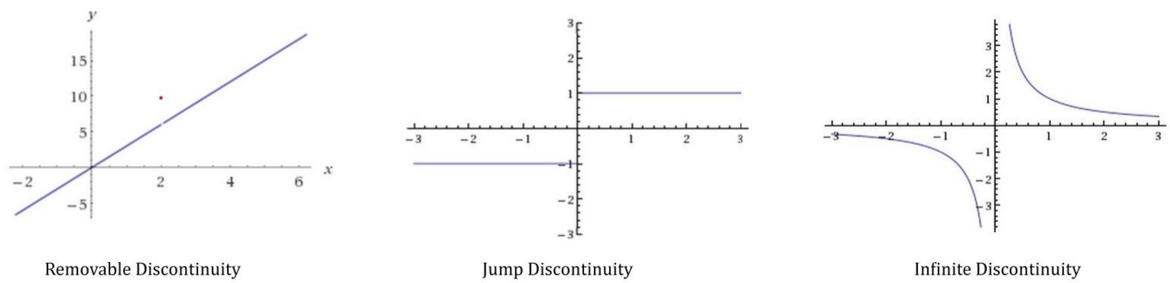


FIGURE 2.0.2. The three types of discontinuities

Polynomial functions and exponential function are continuous on $(-\infty, \infty)$. Rational functions, log functions, and trigonometric functions are continuous on their domains (keeping in mind that the domains of rational functions and trigonometric functions are interrupted by vertical asymptotes).

EXAMPLE 2.2. Suppose $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \leq 2 \\ x^3 + 1 & \text{if } x > 2 \end{cases}$. At what points is $f(x)$ discontinuous. Identify the type discontinuity.

Solution. The graph of $f(x)$ is shown in Figure 2.0.3. [2] Observe that there is a jump discontinuity at 2 and infinite discontinuity at 0.

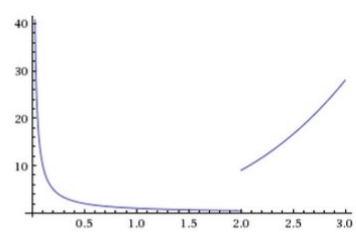


FIGURE 2.0.3. The graph of the piecewise function in 2.2

The concept of continuity leads to our first theorem in Calculus called the Intermediate Value Theorem. We will understand it intuitively here and prove it in Chapter 12.

[2]Enter `Piecewise[1/x, x<= 2, x caret symbol 3+1, x>2] x from -1 to 3`

Theorem 2.3. (Intermediate Value theorem) Suppose f is a real-valued function that is continuous on $[a, b]$ and suppose w is a number such that $f(a) \leq w \leq f(b)$. Then, there exists $c \in [a, b]$ such that $f(c) = w$.

According to the theorem, when the function is continuous, we can choose any number w between $f(a)$ and $f(b)$ on the y-axis and corresponding to that number w there exists a number c between a and b on the x-axis such that $f(c) = w$ (See Figure 2.0.4). The theorem does not tell us how to find the number c, just that there exists such a number. Such results are called **Existence Theorems**. At this stage it is enough to understand the proof of the Intermediate Value Theorem intuitively. The proof is in Chapter 12.

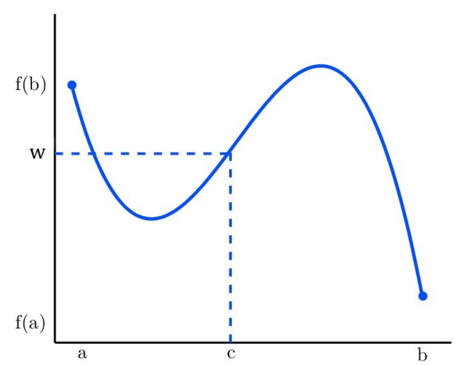


FIGURE 2.0.4. Picture for Theorem 2.3

The converse is not true. For example, consider the function:

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

For any interval $[a, b]$ containing 0, $f(x)$ takes on every value between $f(a)$ and $f(b)$ somewhere in $[a, b]$. But f is not continuous at 0, because $\lim_{x \to 0} f(x)$ does not exist.

As a corollary we obtain the following result whose proof is easy. A **root** of a function f is a number c such that $f(c) = 0$. Roots are also called **zeros** of the function.

Corollary 2.4. (Bolzano's Theorem) Suppose f is a function that is continuous on $[a, b]$ and suppose $f(a)$ and $f(b)$ have opposite signs. Then there exists a root of f in $[a, b]$.

According to the corollary, if $f(a)$ is negative and $f(b)$ is positive and we can draw the function without lifting pencil from paper, then at some point the pencil will cross the x-axis (See Figure 2.0.5). That is precisely where a root or zero of the function occurs.

The proof of the above result follows easily from the Intermediate Value Theorem. Since f is continuous and $f(a)$ and $f(b)$ have opposite signs, the number 0 lies between $f(a)$ and $f(b)$. By the Intermediate Value Theorem, there is a number c in $[a, b]$ such that $f(c) = 0$.

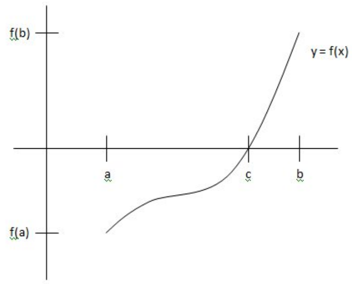


Figure 2.0.5. Picture for Corollary 2.4

This number c is a root of f. However, we will see in Chapter 12 that this result is a step in the proof of the Intermediate Value Theorem.

We will end this chapter with an excerpt from Bertrand Russell's essay "Mathematics and the Metaphysicians," [3] Historically, limits and continuity came after the discovery of Calculus, while mathematicians were trying to make Calculus rigorous. Subsequently, they are presented first for pedagogical reasons, so concepts follow in logical order. Unfortunately this has the effect of discouraging students because limits is harder to grasp than derivatives and integrals. Moreover, the presentation of limits in many textbooks is made unnecessarily hard by dwelling on formality. Those who find limits challenging should find it reassuring to know that limits were hard for mathematicians too.

> Calculus required continuity, and continuity was supposed to require the infinitely little; but nobody could discover what the infinitely little might be. It was plainly not quite zero, because a sufficiently large number of infinitesimals, added together, were seen to make up a finite whole. But nobody could point out any fraction which was not zero, and yet not finite. Thus there was a deadlock. But at last Weierstrass discovered that the infinitesimal was not needed at all, and that everything could be accomplished without it. Thus there was no longer any need to suppose that there was such a thing. ...
>
> The banishment of the infinitesimal has all sorts of odd consequences, to which one has to become gradually accustomed. For example, there is no such thing as the next moment. The interval between one moment and the next would have to be infinitesimal, since, if we take two moments with a finite interval between them, there are always other moments in the interval. Thus if there are to be no infinitesimals, no two moments are quite con- secutive, but there are always other moments between any two. Hence there must be an infinite number of moments between any two ; because if there were a finite number one would be nearest the first of the two moments, and therefore next to it.

[3] Bertrand Russell's book of essays titled *Mysticism and Logic* is available online at https://archive.org/details/mysticismlogicot00russ. This particular essay begins on page 74.

CHAPTER 3

Techniques for Finding Limits

In the previous chapter we found limits visually, but we also saw that it is not always the easiest way nor is it always reliable. In this chapter we learn some techniques for finding limits beginning with the limit rules and formulas. The proofs require the formal epsilon-delta definition of limits. They are given in Chapter 12. The distinction between a forumula and a rule is minor. A formula applies to a specific function and a rule applies to any real-valued function.

Theorem 3.1. (Limit Formulas) Let a and c be real numbers and n be a natural number.

(1) $\lim_{x \to a} c = c$

(2) $\lim_{x \to a} x = a$

(3) $\lim_{x \to a} x^n = a^n$

(4) $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$

Theorem 3.2. (Limit Rules) Let $f(x)$ and $g(x)$ be two real-valued functions, let a be a real number, and let n be a natural number.

(1) $\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)]$

(2) $\lim_{x \to a} f(x)g(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$

(3) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

(4) $\lim_{x \to a}[f(x)]^n = [\lim_{x \to a} f(x)]^n$

(5) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$, where if n is even, then is a positive natural number (otherwise we get complex numbers).

For example, using these Formulas (1) and (2) and Rules (1) and (2) we can show that

$$\lim_{x \to 8} 2x + 5 = \lim_{x \to 8} 2x + \lim_{x \to 8} 5 = \lim_{x \to 8} 2 \lim_{x \to 8} x + \lim_{x \to 8} 5 = 2x + 8$$

But this sort of tedious applications of the rules and formulas is not how we will approach limit problems. For this example, it makes sense to use the fact that a linear function is a continuous function. Recall that we defined a continuous function as a function $f(x)$ with the property that $\lim_{x\to a} f(x) = f(a)$. So for continuous functions finding the limit at a point a amounts to finding the value of the function at a (assuming a is in the domain). Functions like polynomial functions, rational functions, radical functions, exponential functions, and log functions are continuous on their domains. We will use this feature to obtain limits.

Example 3.3. The following functions are continuous on their domains. Use this fact to find the limits.

(1) $\lim_{x\to 1} x^5 + 2x^3 - 3x^2 + 4x - 5$

(2) $\lim_{x\to 3} \frac{2x^2+x-5}{x-1}$

(3) $\lim_{x\to -2} 3^x - 5$

(4) $\lim_{x\to -2} (3^x - 1)^2$

(5) $\lim_{x\to 16} \sqrt{x} + 5$

(6) $\lim_{x\to -27} \sqrt[3]{x} + 1$

Solutions.

(1) $\lim_{x\to 1} x^5 + 2x^3 - 3x^2 + 4x - 5 = 1 + 2 - 3 + 4 - 5 = -1$

(2) $\lim_{x\to 3} \frac{2x^2+x-5}{x-1} = \frac{2(3^2)+3-5}{3-1} = \frac{16}{2} = 8$

(3) $\lim_{x\to -2} 3^x - 5 = 3^{-2} - 5 = \frac{1}{9} - 5 = \frac{-44}{9}$

(4) $\lim_{x\to -2} (3^x - 1)^2 = (3^{-2} - 1)^2 = (\frac{1}{9} - 1)^2 = (\frac{-8}{9})^2 = \frac{64}{81}$

(5) $\lim_{x\to 16} \sqrt{x} + 5 = \lim_{x\to 16} \sqrt{16} + 5 = 4 + 5 = 9$

(6) $\lim_{x\to -27} \sqrt[3]{x} + 1 = \lim_{x\to -27} \sqrt[3]{-27} + 1 = -3 + 1 = -2$

It isn't hard to use a visual understanding of limits for these functions, but using the notion of continuous functions is much more elegant. This is a short list of a large number of limit problems that can be solved simply by recognizing that the function is continuous on its domain (and the point a is in the domain). Pay careful attention to rational functions which are continuous only on their domains. The roots of the denominator are not in the domain. This is where the vertical asymptotes occur. For example, since the domain of $f(x) = \frac{2x^2+x-5}{x-1}$ is all real numbers except 1, the function is continuous everywhere else. [1]

Rational functions can have holes in them that are hidden at first glance as in the next example. In fact, when finding the limit of a rational function we should first check for holes by checking if we get $\frac{0}{0}$. Later on, after we study derivatives we will learn a method called L'Hopital's rule that will work in all situations that give $\frac{0}{0}$. But some rational functions can be handled quite easily without a fancier technqiue.

Example 3.4. Find $\lim_{x\to 1} \frac{x^2-1}{x-1}$.

Solution. Observe that 1 is a root of the denominator. Check to see if it is also a root of the numerator giving the $\frac{0}{0}$ form.

$$\frac{x^2-1}{x-1} = \frac{0}{0}.$$

Thus we must first factor out $(x-1)$ and get

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = x+1 = 1+1 = 2.$$

□

Note that we can cancel $(x-1)$ from numerator and denominator because we are saying x tends to 1. We are not saying $x = 1$. The graph of $f(x) = \frac{x^2-1}{x-1}$ is the straight line $y = x+1$ with a hole at 1.

How would we tackle a limit problem involving a rational function where the denominator is a high degree polynomial? For example, consider the function $f(x) = \frac{x^3-x^2+x+5}{x^7+x^4-x^3+3x^2-4}$ where factoring the denominator to find the zeros would be difficult, to say the least, and possibly outside the scope of a Calculus course if the Rational Root Test does not work. Even if the Rational Root Test does work it would be unbearably tedious to solve a 7th degree equation.

Example 3.5. Find $\lim_{x\to 2} \frac{x^3-x^2+x+5}{x^7+x^4-x^3+3x^2-4}$.

Solution. Check to confirm that 2 is not a root of the polynomial in the denominator $x^7+x^4-x^3+3x^2-4$. Plugging $x = 2$ in it gives $2^7+2^4-2^3+3(2)^2-4 = 192$. So 2 is not a root of the denominator, and therefore in the domain of the rational function. Now the problem is just like the previous ones.

$$\lim_{x\to 2} \frac{x^3-x^2+x+5}{x^7+x^4-x^3+3x^2-5} = \frac{2^3-2^2+2+5}{2^7+2^4-2^3+3(2)^2-4} = \frac{11}{192}$$

□

[1] The rest of this chapter may be omitted by students taking a "Survey of Calculus" course

Let's do the same problem again taking the limit value as 1 instead of 2. So the problem is to find

$$\lim_{x\to 1} \frac{x^3 - x^2 + x + 5}{x^7 + x^4 - x^3 + 3x^2 - 4}.$$

Observe that

$$\frac{x^3 - x^2 + x + 5}{x^7 + x^4 - x^3 + 3x^2 - 4} = \frac{1^3 - 1^2 + 1 + 5}{(1)^7 + (1)^4 - (1)^3 + 3(1)^2 - 4} = \frac{6}{0}.$$

So 1 is a root of the polynomial in the denominator, but not a hole since 1 is not a root of the polynomial in the numerator. Thus, there is a vertical asymptote at 1. The function may go to ∞ on both sides of the point 1, or it may go to $-\infty$, or it may go to ∞ on one side and $-\infty$ on the other. We cannot tell with the methods we have. The best we can say is the limit is ∞ or $-\infty$ or does not exist.

To summarize, suppose a number a is not a hole of a rational function (checked by verifying we don't get $\frac{0}{0}$), but a root of the denominator, then we can narrow the answer to three choices:

- "does not exist" if the function approached ∞ on one side of the zero and $-\infty$ on the other size;
- ∞ if on either side of the root the function approached ∞.
- $-\infty$ if on either side of the root the function approached $-\infty$.

There is something unsatisfying about saying the answer is one of three possibilities. That is why we look for more and more sophisticated methods. Incidently, the graph of $f(x) = \frac{x^3-x^2+x+5}{x^7+x^4-x^3+3x^2-4}$ in shown in Figure 3.0.1. Observe that

$$\lim_{x\to 1} \frac{x^3 - x^2 + x + 5}{x^7 + x^4 - x^3 + 3x^2 - 4} \text{ dne}$$

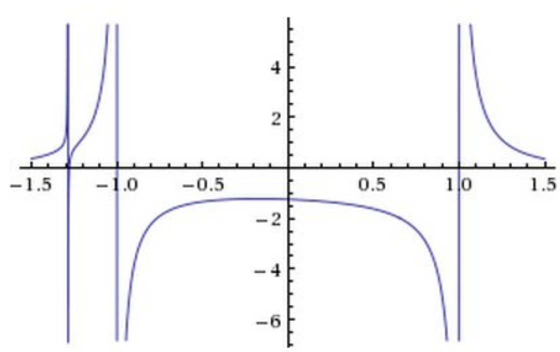


FIGURE 3.0.1. $f(x) = \frac{x^3-x^2+x+5}{x^7+x^4-x^3+3x^2-4}$

The hardest situation would occur if the rational function had high degree polynomials in the numerator and denominator and the number a turned out to be a hole. Then we would have to divide the numerator and denominator by $(x-a)$, cancel out the common factor $(x-a)$ and repeat all over again. Fortunately, later on we will learn L'Hopital's Rule that can solve the problem in less than a minute.

The second technique we study is an algebraic way of handling rational functions with radicals without graphing them. This technique is called "Rationalizing the denominator." To implement it multiply numerator and denominator by the rational conjugate.

Example 3.6. Find $\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2}$.

Solution.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2} &= \lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2} \times \frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3} \\
&= \lim_{x\to 0} \frac{x^2+9-9}{x^2\sqrt{(x^2+9)}+3} \\
&= \lim_{x\to 0} \frac{x^2}{x^2\sqrt{(x^2+9)}+3} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{(x^2+9)}+3} \\
&= \frac{1}{6}
\end{aligned}$$

Notice how algebraic operations can simplify the function so that plugging 0 in it does not cause problems in the denominator. This method will not work for all radical expressions, but when it works it works well.

The third technique is a more sophisticated method that depends on a theorem. The proof of this theorem is given in Chapter 12.

Theorem 3.7. (Sandwich Theorem) suppose $f(x) \le h(x) \le g(x)$, for every x in an open interval containing a point a except possibly at a. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$, then $\lim_{x\to a} h(x) = L$.

See Figure 3.0.2 to understand this result intuitively. Since $f(x) \le h(x) \le g(x)$, the limit gets trapped between $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ and since both are L, it follows that $\lim_{x\to a} h(x)$ is also L. Hence the name Sandwich Theorem.

Example 3.8. Find $\lim_{x\to 0} x^2 \sin \frac{1}{x^2}$.

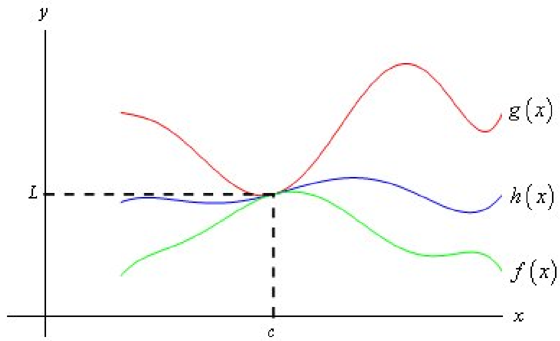


FIGURE 3.0.2. The sandwich Theorem

Solution. Our strategy is to find two functions that sandwich $x^2 \sin \frac{1}{x^2}$. We do this in a clever manner using the fact that $\sin x$ lies between -1 and 1.

$$-1 \leq \sin \frac{1}{x^2} \leq 1$$

$$-x^2 \leq \sin \frac{1}{x^2} \leq x^2$$

$$\lim_{x \to 0}(-x^2) \leq x^2 \sin \frac{1}{x^2} \leq \lim_{x \to 0}(x^2)$$

Since $\lim_{x \to 0} -x^2 = 0$ and $\lim_{x \to 0} x^2 = 0$, by the Sandwich Theorem, $\lim_{x \to 0} x^2 \sin \frac{1}{x^2} = 0$.

These problems can be tricky and require a lot of practice.

The fourth technique is specifically for finding the limit of a rational function as x tends to ∞ or $-\infty$ without graphing it. Here we rely on finding the horizontal asymptote using the Horizontal Asymptote Theorem

Theorem 3.9. (Horizontal Asymptote Theorem) Suppose

$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots b_1 x + b_0}$$

is a rational function with $a_n, b_m \neq 0$.

(1) If $n = m$, then the horizontal asymptote is $y = \frac{a_n}{b_m}$ and $\lim_{x \to \infty} f(x) = \frac{a_n}{b_m}$;

(2) If $n < m$, then the x-axis is the horizontal asymptote and $\lim_{x \to \infty} f(x) = 0$; and

(3) If $n > m$, then there is no horizontal asymptote.

Like other techniques the Horizontal Asymptote Theorem has its limitations. It works very well for Case (i) and (ii). It doesn't work for Case (iii) because there are slant asymptotes and $\lim_{x \to \infty} f(x)$ may be ∞ or $-\infty$.

EXAMPLE 3.10. Find the limits.

(1) $\lim_{x\to\infty} \frac{3x+4}{5x+7}$

(2) $\lim_{x\to\infty} \frac{2x^2-5}{3x^2+x+5}$

(3) $\lim_{x\to\infty} \frac{x^2-1}{x^2+1}$

(4) $\lim_{x\to\infty} \frac{x^2+3x+1}{3x^4-1}$

Solutions.

(1) $\lim_{x\to\infty} \frac{3x+4}{5x+7} = \frac{3}{4}$

Note that, since the degree of the numerator is the same as the degree of the denominator, we can give the answer instantly using the Horizontal Asymptote Theorem.

(2) $\lim_{x\to\infty} \frac{2x^2-5}{3x^2+x+5} = \frac{2}{3}$

(3) $\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1$

(4) $\lim_{x\to\infty} \frac{x^2+3x+1}{3x^4-1} = 0$

Note that since the degree of the numerator is less than the degree of the denominator, by the Horizontal Asymptote Theorem, the x-axis, i.e. the line $y = 0$, is the horizontal asymptote.

□

The ease of answering these limit problems makes the Horizontal Asymptote Theorem a powerful technique and worth understanding thoroughly. So what exactly is going on here? Let's take another look at the first example

$$\lim_{x\to\infty} \frac{3x+4}{5x+7}$$

Divide numerator and denominator by the highest power of x, in this case just x, to get

$$\begin{aligned}
\lim_{x\to\infty}\frac{3x+4}{5x+7} &= \lim_{x\to\infty}\frac{\frac{3x+4}{x}}{\frac{5x+7}{x}}\\
&= \lim_{x\to\infty}\frac{\frac{3x}{x}+\frac{4}{x}}{\frac{5x}{x}+\frac{7}{x}}\\
&= \lim_{x\to\infty}\frac{3+\frac{4}{x}}{5+\frac{7}{x}}\\
&= \frac{\lim\limits_{x\to\infty}3+\lim\limits_{x\to\infty}\frac{4}{x}}{\lim\limits_{x\to\infty}5+\lim\limits_{x\to\infty}\frac{7}{x}}\\
&= \frac{3+0}{5+0}\\
&= \frac{3}{5}
\end{aligned}$$

Consider the fourth example again

$$\lim_{x\to\infty}\frac{x^2+3x+1}{3x^4-1}$$

Divide numerator and denominator by the highest power of x, in this case x^4, to get

$$\begin{aligned}
\lim_{x\to\infty}\frac{x^2+3x+1}{x^4-1} &= \lim_{x\to\infty}\frac{\frac{x^2}{x^4}+\frac{3x}{x^4}+\frac{1}{x^4}}{\frac{x^4}{x^4}-\frac{1}{x^4}}\\
&= \lim_{x\to\infty}\frac{\frac{1}{x^2}+\frac{3}{x^3}+\frac{1}{x^4}}{3-\frac{1}{x^4}}\\
&= \frac{\lim\limits_{x\to\infty}\frac{1}{x^2}+\lim\limits_{x\to\infty}\frac{3}{x^3}+\lim\limits_{x\to\infty}\frac{1}{x^4}}{\lim\limits_{x\to\infty}3-\lim\limits_{x\to\infty}\frac{1}{x^4}}\\
&= \frac{0}{3}\\
&= 0
\end{aligned}$$

These are all the steps where we carefully apply the limit rules and get the answer. We are expected to know the graph of $y=\frac{1}{x}$ and the fact that $\lim\limits_{x\to\infty}\frac{1}{x}=0$. We will prove this formally in Chapter 12.

Next we will give a proof of the first two cases of the Horizontal Asymptote Theorem. The proofs are just like the steps in the above two problems, except that we have arbitrary polynomials in the numerator and denominator instead of specific ones. The third case requires the formal definition of limits and appears in Chapter 12.

Partial Proof of the Horizontal Asymptote Theorem.

(1) Suppose $n = m$.

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \dots b_1 x + b_0}.$$

Since the highest power of x is n, divide numerator and denominator by x^n to get

$$f(x) = \frac{\frac{a_n x^n}{x^n} + \frac{a_{n-1}x^{n-1}}{x^n} + \dots \frac{a_1 x}{x^n} + \frac{a_0}{x^n}}{\frac{b_n x^n}{x^n} + \frac{b_{n-1}x^{n-1}}{x^n} + \dots \frac{b_1 x}{x^n} + \frac{b_0}{x^n}}$$

Since limit of a quotient is quotient of a limit and limit of a sum is sum of limits, we can take limit of each term. As x tends to ∞ all the terms tend to 0 except the first term in the numerator and the first term in the denominator where the x^n cancel out. So

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{a_n}{b_n} = \frac{a_n}{b_n}.$$

(2) Suppose $n < m$. Then

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \dots b_1 x + b_0}.$$

Since the highest power of x is m, divide numerator and denominator by x^m to get

$$f(x) = \frac{\frac{a_n x^n}{x^m} + \frac{a_{n-1}x^{n-1}}{x^m} + \dots \frac{a_1 x}{x^m} + \frac{a_0}{x^m}}{\frac{b_m x^m}{x^m} + \frac{b_{m-1}x^{m-1}}{x^m} + \dots \frac{b_1 x}{x^m} + \frac{b_0}{x^m}}$$

Since limit of a quotient is quotient of a limit and limit of a sum is sum of limits, we can take limit of each term. As x tends to ∞ all the terms tend to 0 except the first term in the denominator where the x^m cancels out. So

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{0}{b_m} = 0$$

□

The technique of dividing numerator and denominator by the highest power of x is very useful even when the function is not a rational function.

Example 3.11. Find the limits.

(1) $\lim\limits_{x\to\infty} \frac{\sqrt{10x^2+2}}{4x+3}$

Solution. Begin by dividing by the highest power of x. Since x^2 is under the radical sign the highest power of x is x itself.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{10x^2+3}}{4x+3} &= \lim_{x\to\infty} \frac{\sqrt{x^2(10+\frac{2}{x^2})}}{4x+3} \\
&= \lim_{x\to\infty} \frac{\frac{x\sqrt{10+\frac{2}{x^2}}}{x}}{\frac{4x+3}{x}} \\
&= \lim_{x\to\infty} \frac{\sqrt{10+\frac{2}{x^2}}}{4+\frac{3}{x}} \\
&= \frac{\sqrt{10}}{4}
\end{aligned}$$

□

This brings us to the end of Chapter 3. You should think of the techniques for solving limits as a set of tools and like any set of tools, the same technique doesn't work for all the limit problems. With practice it becomes easier to recognize which ones work for which problems.

Practice Problems

(1) Find the limits using the limit rules.

a) $\lim_{x\to\sqrt{2}} 15$
b) $\lim_{x\to -2} x$
c) $\lim_{x\to 4} 3x-4$
d) $\lim_{x\to -2} \frac{x-5}{4x+3}$
e) $\lim_{x\to 1} (-2x+5)^4$
f) $\lim_{x\to -2} (3x^3-2x+7)$
g) $\lim_{x\to 4} \frac{6x-1}{2x-9}$
h) $\lim_{x\to 2} \frac{x^2+x-2}{(x-2)^2}$
i) $\lim_{x\to -2} \frac{x^3+8}{x^4-16}$
j) $\lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2}$
k) $\lim_{x\to 1} \left(\frac{x^2}{x-1}-\frac{1}{x-1}\right)$
l) $\lim_{x\to 0} \frac{4-\sqrt{x+16}}{x}$

m) $\lim_{x\to 1} \frac{x^2-x-2}{x^5-1}$
n) $\lim_{x\to 3} x^2(3x-4)(9-x^3)$
o) $\lim_{x\to 5^+} \sqrt{x^2-25}+3$
p) $\lim_{x\to 3^+} \frac{\sqrt{(x-3)^2}}{x-3}$

(2) Use the Sandwich Theorem to find the following limits. Show your work in detail.

a) If $4x-9 \le f(x) \le x^2-4x+7$ for $x \ge 0$, find $\lim_{x\to 4} f(x)$.
b) If $2x \le g(x) \le x^4-x^2+2$ for all x, find $\lim_{x\to 1} g(x)$.
c) $\lim_{x\to 0} \sqrt{x^3+x^2}\sin\frac{\pi}{x}$
d) $\lim_{x\to 0} x^4 \cos\frac{2}{x}$

(3) Find the limit, if it exists, using a technique from this chapter.

a) $\lim_{x\to 5^-} \sqrt{5-x}$
b) $\lim_{x\to 2} \sqrt{8-x^3}$
c) $\lim_{x\to 1} \sqrt[3]{x^3-1}$
d) $\lim_{x\to 4} \frac{5}{x-4}$
e) $\lim_{x\to \frac{5}{2}} \frac{8}{(2x+5)^3}$
f) $\lim_{x\to \infty} \frac{1}{x(x-3)^2}$
g) $\lim_{x\to \infty} \frac{-x^3+2x}{2x^2-3}$
h) $\lim_{x\to -\infty} \frac{2-x^2}{x+3}$
i) $\lim_{x\to -\infty} \frac{4x-3}{\sqrt{x^2-1}}$
j) $\lim_{x\to \infty} \frac{2x^2}{x^2-x-2}$
k) $\lim_{x\to \infty} \frac{5x^2-3x+1}{2x^2+4x-7}$
l) $\lim_{x\to \infty} \frac{3x}{(x+8)^2}$
m) $\lim_{x\to -\infty} \frac{4-7x}{2+3x}$
n) $\lim_{x\to -\infty} \frac{2x^2-3}{4x^3+5x}$
o) $\lim_{x\to \infty} \sqrt[3]{\frac{8+x^2}{x(x+1)}}$

CHAPTER 4

What is a Derivative?

German mathematician Gottfried Wilhelm von Leibniz (1646 - 1716) and British mathematician Sir Isaac Newton (1642 - 1726) get the credit for discovering Calculus. Poet Alexander Pope beautifully captures the greatness of this discovery as follows:

> Nature and nature's laws lay hid in night:
>
> God said, "Let Newton be!" and all was light.

Newton and Leibniz discovered Calculus independently and the ensuing fight for credit makes for interesting reading (see Figure 4.0.1). [1]

Sir Isaac Newton   Gottfried Wilhelm Leibniz

FIGURE 4.0.1. The fathers of Calculus

Suppose we have to drive from Brooklyn, NY to Niagara Falls, NY. Google shows that the distance is 412 miles and it takes 7 hours. Using the well-known formula

$$distance = speed \times time$$

we may conclude that our speed is

$$speed = \frac{distance}{time} = \frac{412}{7} = 58.86 \text{ miles/hour.}$$

[1]Images taken from Wikipedia. This portrait of a 46 year old Newton was painted in 1689 by famous potrait artist Godfrey Kneller. Leibniz's portrait appears in the Public Library of Hannover.

Does this mean we were driving exactly 58.86 miles per hour the entire way? A glance at the odometer shows that the needle is constantly moving. So the "average speed" may be 58.86 miles per hour over the entire 7 hour trip, but "instantaneous speed" is changing constantly. And what exactly is an instant? Is it 1 minute, 1 second, 0.1 second, or smaller? We represent an instant as a point on the number line. A point is an infinitessimily small 0-dimensional entity. In other words it has no length, breadth, or height. Note that we cannot actually draw a point that has no length, breadth, or height, no matter how fine we make it, so we must imagine that the dot on the page has no dimension.

There is an inherent difficulty capturing the concept of motion at an instant. Arguably at an instant there is no motion because if we take a photo of a moving car we will get a picture of a motionless car. Zeno of Elea (c. 490 BC - 430 BC) was grappling with these concepts in the 3rd century BC. He was a member of the Eleatic School founded by the philosopher Parmenides (c. 515 BC - 460 BC), who is famous for his poem on the impermeance of change. [2]

> Mortals have made up their minds to name two forms, one of which they should not name, and that is where they go astray from the truth. They have distinguished them as opposite in form, and have assigned to them marks distinct from one another. To the one they allot the fire of heaven, gentle, very light, in every direction the same as itself, but not the same as the other. The other is just the opposite to it, dark night, a compact and heavy body. Of these I tell you the whole arrangement as it seems likely; for so no thought of mortals will ever outstrip you.

Zeno wrote a set of three paradoxes that appear in Aristotle's *Physics*. We will dicuss the first two later; the third is relevant at this junction.

> If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless.

In this "flying arrow is motionless" paradox, Zeno is trying to make arguments in support of Parmenides's doctrine that our senses cannot be trusted; change, and motion in particular, is just an illusion. Zeno was arguing that an object "occupies" a small space at an instant, and is therefore motionless at an instant. Aristotle rejected Zeno's reasoning in no uncertain terms.

> Zeno's reasoning, however, is fallacious, when he says that if everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is

[2] Parmenides wrote a poem *On Nature* explaining his philosophy. The first part called *Proem* is the introduction, the second called *The Way of Truth* gives his view on reality, and the third called The Way of Opinion explains the deceptive nature of reality as it appears to us. The complete version has not been found, but other philosophers were greatly influenced by it and wrote extensively about it. It somewhat mind-bending to read, the poem is full of interesting philosophical ideas. See http://www.mycrandall.ca/courses/grphil/parmenides.htm and http://plato.stanford.edu/entries/parmenides/ for more information.

> therefore motionless. This is false, for time is not composed of indivisible moments any more than any other magnitude is composed of indivisibles.

However, Zeno has the last laugh, and if Aristotle had not so resolutely rejected his attempts to understand motion, perhaps Calculus might have been invented earlier.

The fresco in Figure 4.0.2 shows Plato and Aristotle in the center under the arch, Euclid on the right holding a compass, Zeno on the far left, Phythagorous reading a book, and Parmenides in an orange robe looking down at him. It was drawn in 1511 by the famous painter Raphael in the Sistine Chapel at the Vatican, Italy [3]

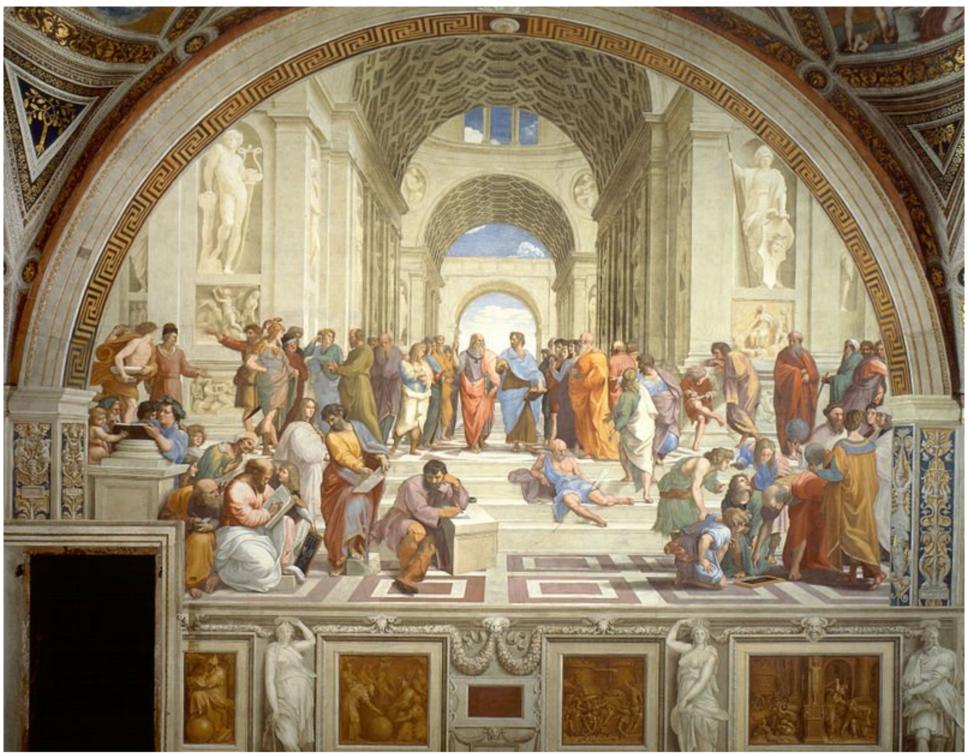


FIGURE 4.0.2. School Of Athens

There is some paradox in trying to quantify motion at an instant because when the focus is on a single moment the motion stops. Sir Issac Newton's and independently Gottfried Wilhelm Leibniz's brilliant approach to motion was to stop looking for a way to define motion at an instant, but instead to look at smaller and smaller intervals containing that instant. Consider Figure 4.0.3 that gives the distance the car every hour for the 7 hour trip from Brooklyn to Niagara Falls. The overall distance covered is 412 hours in 7 hours. As noted earlier the average speed is $\frac{412}{7} = 58.86$ miles per hour. The graph on the right of the table is a plot of time on the x-axis and distance on the y-axis.

Consider a different scenario: throw an apple straight up in the air. Suppose it hits the ground in 7 seconds. The table in Figure 4.0.4 gives the height of the apple at each second. The graph next to it is the graph of time versus height of the orange (not to be confused with the path the apple takes). Observe that the apple starts at 6 feet assuming

[3] Image taken from Wikipedia

Time (hours)	Distance (miles)
0	0
1	30
2	80
3	155
4	220
5	280
6	355
7	412

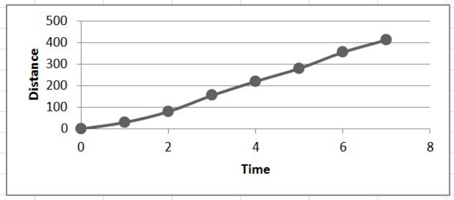


FIGURE 4.0.3. Distance traveled by a car

the person throwing the apple up is 6 feet tall. It reaches a highest point of 50 feet after which it falls back down reaching the ground in 7 seconds. What now is the average height since technically at the end the apple has height 0? The speed in this scenario has direction. When the apple is going up the speed is positive and when it is coming down the speed is negative. Speed with direction (positive or negative) is called **velocity**. From now on we will use the term velocity.

Time (seconds)	Height (feet)
0	6
1	10
2	35
3	50
4	40
5	25
6	15
7	0

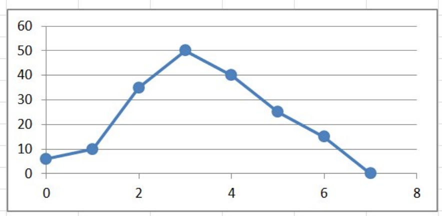


FIGURE 4.0.4. Height of an apple thrown up

Instead of trying to define velocity at an instant of time, consider a small interval (a, b) around an instant t and define average velocity as follows:

$$\text{Average velocity at } t = \frac{\text{change in distance}}{\text{change in time}} = \frac{f(b) - f(a)}{b - a}..................(1)$$

Observe that $\frac{f(b)-f(a)}{b-a}$ is the slope of the line that passes through the points $(a, f(a))$ and $(b, f(b))$. It is called the **secant line**. If we let

$$b - a = h,$$

then

$$b = a + h.$$

We can write (1) as

$$\text{Average velocity at } t = \frac{f(a+h) - f(a)}{h}.................(2)$$

As the interval shrinks, in other words as h tends to 0, the point $a+h$ merges with the point a and we can write (2) as

$$\text{Instantaneous velocity at } a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This is the **derivative at** a. The secant line becomes the **tangent line at** a.

There is nothing special about distance, time, and velocity (other than this is how it all started). We can define this concept for any function. Let f be any continuous function defined on an open interval (a, b). Consider the secant line joining points $(a, f(a))$ and $(b, f(b))$. The slope of this secant line is

$$\text{slope} = \frac{f(b) - f(a)}{b - a}.$$

Let $b = a + h$ and write the interval as $(a, a + h)$ so the slope can be written as

$$\text{slope} = \frac{f(a + h) - f(a)}{h}.$$

As the interval shinks, that is, h tends to 0, the secant line approaches the tangent line at a as shown in Figure 4.0.5. The slope of the secant line tends to the slope of the tangent line. This is why the geometric interpretation of the derivative is "slope of the tangent line." [4]

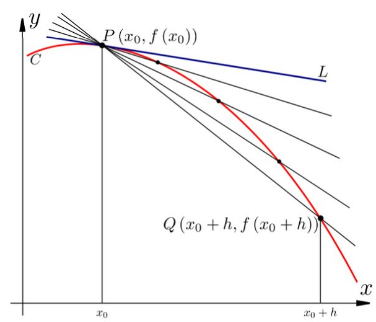


FIGURE 4.0.5. Secant line approaching tangent line

Definition of the Derivative

Let f be a function defined on an open interval containing t. The **derivative of** f **at** t, denoted as $f'(t)$, is defined as

$$f'(t) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h}.$$

If the limit exists at every point t in the domain of f, then f is called a **differentiable function**. In this case the derivative itself may be viewed as a function of f and defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

The derivative is also called the **rate of change** at t (to reflect the slope computation that lies at the heart of the derivative.)

[4] `http://en.wikibooks.org/wiki/Calculus/Differentiation/Differentiation(underscore)Defined`

EXAMPLE 4.1. Find the derivatives of the following functions.

(1) $f(x) = x^2$

(2) $f(x) = x^3$

(3) $f(x) = \sqrt{x}$

(4) $f(x) = \frac{1}{x}$

Solutions.

(1) $f(x) = x^2$

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h\to 0} 2x + h \\ &= 2x \end{aligned}$$

(2) $f(x) = x^3$

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h\to 0} \frac{h(3x^2 + 3h + h^2)}{h} \\ &= \lim_{h\to 0} 3x^2 + 3h + h^2 \\ &= 3x^2 \end{aligned}$$

(3) $f(x) = \sqrt{x}$

$$\begin{aligned} f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\ &= \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

(4) $f(x) = \frac{1}{x}$

$$\begin{aligned} f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} \\ &= \lim_{h\to 0}\frac{\frac{x-(x+h)}{x(x+h)}}{h} \\ &= \lim_{h\to 0}\frac{\frac{h}{x(x+h)}}{h} \\ &= \lim_{h\to 0}\frac{-h}{hx(x+h)} \\ &= \lim_{h\to 0}\frac{-1}{x(x+h)} \\ &= -\frac{1}{x^2} \end{aligned}$$

□

Suppose the function is not known and we just have some data as in Table 4 and we know that x is the independent variable and y is the dependent variable. Then we cannot find the derivative using a formula and we have to adopt a computational approach.

We can compute the slope at each consecutive pairs of points to obtain the approximate derivative at each point. The third row in Table 2 strongly suggests that $f'(2)$ is in the vicinity of 12. We cannot say for sure it is 12, but we had to make an educated guess we would say the derivative at 2 is roughly 12.

The above function is in fact $f(x) = x^3$ and we calculated its derivative precisely earlier as $f'(x) = 3x^2$. Observe that $f'(2) = 3(2^2) = 12$. So there is merit to this computational approach. It is very handy for real-world data that does not fit into a neat well-known

x	$y = f(x)$
1.98	7.762392
1.99	7.880599
2	8
2.01	8.120601
2.02	8.242408

TABLE 1. Data on two related variables x and y

x	y	$\frac{y_2 - y_1}{x_2 - x_1}$
1.98	7.762392	
1.99	7.880599	$\frac{7.880599 - 7.762392}{1.99 - 1.98} = 11.8207$
2	8	$\frac{8 - 7.880599}{2 - 1.99} = 11.9401$
2.01	8.120601	$\frac{8.120601 - 8}{2.01 - 2} = 12.0601$
2.02	8.242408	$\frac{8.242408 - 8.120601}{2.02 - 2.01} = 12.1807$

TABLE 2. Rate of change of y

functions. For example, house prices or the Standard and Poor 500 index cannot be modeled by any function. But we can certainly calculate the derivative or rate of change just based on the data. The next example further illustrates the computational approach.

EXAMPLE 4.2. Find the rate of change of Ph.ds awarded in science and engineering from the data shown in Figure 4.0.6. Interpret your results. Repeat the exercise for non-science and engineering Ph.ds. [5]

Year	Science and engineering	Non-science and engineering
1998	27,274	15,364
1999	25,931	15,166
2000	25,966	15,400
2001	25,529	15,209
2002	24,608	15,417
2003	25,282	15,476
2004	26,274	15,844
2005	27,986	15,395
2006	29,863	15,752
2007	31,800	16,312
2008	32,827	15,975

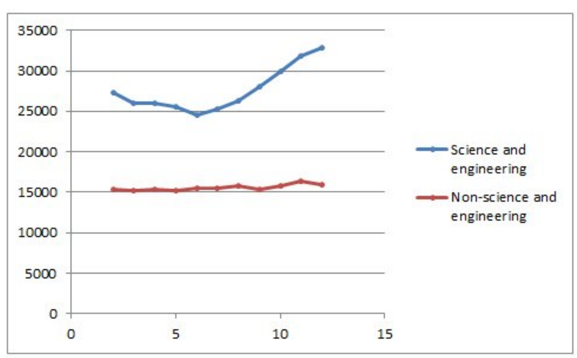


FIGURE 4.0.6. Doctorates awarded: 1998 - 2008

Solution. Observe from Figure 4.0.6 that when the rate of change is negative the graph is decreasing on the interval and when the rate of change is positive the graph is increasing.

Year	Science and engineering	$\frac{y_2-y_1}{x_2-x_1}$
1998	27274	
1999	25931	-1343
2000	25966	35
2001	25529	- 437
2002	24608	-921
2003	25282	674
2004	26274	992
2005	27986	1712
2006	29863	1877
2007	31800	1937
2008	32827	1027

TABLE 3. Rate of change of doctorates awarded

From 2003 to 2007 the rate of change was increasing, but in 2008 the rate of change decreased (even though the number of doctorates awarded is 1027 more than those awarded in 2007). □

Next, we will look at an example where the derivative does not exist. But first let's layer in two more terms: the left derivative and the right derivative. We define them in a manner similar to left and right limits.

Definition of Left and Right Derivative

The left derivative of f at t, denoted as $f'^-(t)$ is defined as

$$f'^-(t) = \lim_{h\to 0^-} \frac{f(t+h)-f(t)}{h}.$$

The right derivative of f at t, denoted as $f'^+(t)$ is defined as

$$f'^+(t) = \lim_{h\to 0^+} \frac{f(t+h)-f(t)}{h}.$$

If the left and right derivative both exist and are equal then the derivative exists and is equal to the left (and right) derivative.

Calculating left and right derivatives and showing they are unequal is a way of showing the derivative does not exist.

EXAMPLE 4.3. Show that the derivative of $f(x) = |x|$ does not exist at 0.

Solution.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|-0}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

The function $y = \frac{|x|}{x}$ is the same as the step function

$$y = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Revisiting its graph shown in Figure 1.0.3 we see that

$$\lim_{x\to 0} \frac{|x|}{x} \text{ dne}$$

since the left and right limits are not the same. Therefore, $f'(0)$ does not exist. □

Informally, we can recognize points where the derivative does not exist by looking at the graph of $f(x) = |x|$ itself. See the left graph in Figure 4.0.7. Let us try zooming in near 0 (graph on the right in Figure 4.0.7). Observe that the zoomed in graph on the right looks exactly the same as the original graph on the left. In fact, the only indication that these are not the exact same figures appears on the x-axis. No matter how much we zoom in the sharp point at 0 will be present in $f(x) = |x|$.

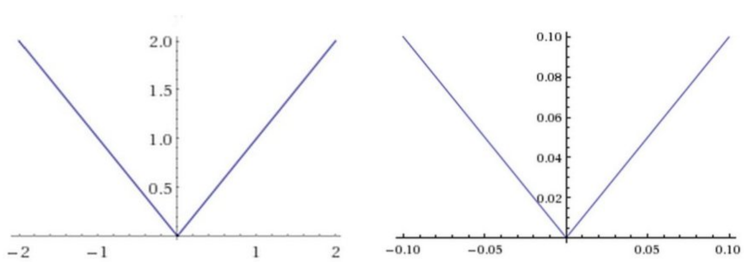


FIGURE 4.0.7. $f(x) = |x|$

Constrast this with the function $f(x) = x^2$ where as you zoom in closer and closer around 0 the curve will look like more and more like a straight line. In fact, this goes to the heart of the derivative notion as slope of the tangent line at a point. In a "smooth" curve like any polynomial function, if you zoom in at a point it will look a straight line. But for a function like $f(x) = |x|$ with a sharp point at 0, there is an abrupt change occuring. The function is not smooth. [6] The next example is another example of a function where the derivative does not exist and in this case the reason is quite different.

EXAMPLE 4.4. Show that the derivative of $f(x) = x^{\frac{1}{3}}$ does not exist at 0.

[6]Smooth is a technical term that means the derivatives of all orders exist. We will discuss it later.

Solution.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^{\frac{1}{3}}-0}{h}$$
$$= \lim_{h\to 0} \frac{h^{\frac{1}{3}}}{h}$$
$$= \lim_{h\to 0} h^{\frac{-2}{3}}$$

From the graph of $y = x^{\frac{-2}{3}}$ shown in Figure 4.0.8, we see that $\lim_{h\to 0} h^{\frac{-2}{3}} = \infty$. Therefore $f'(0)$ does not exist. $\square$

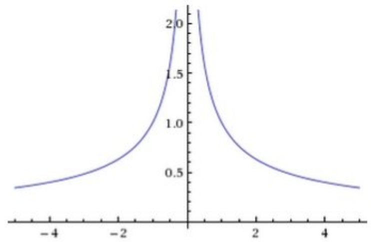


FIGURE 4.0.8. $y = x^{\frac{-2}{3}}$

As in the case of the previous example, we can determine this just by looking at the graph of $f(x) = x^{\frac{1}{3}}$ shown in Figure 4.0.9. Observe that the tangent line at 0 becomes vertical. The slope of a vertical line does not exist. [7] Corner points, vertical tangent lines, and a few more visual cues like a "cusp" shown in Figure 4.0.10 indicate the derivative does not exist.

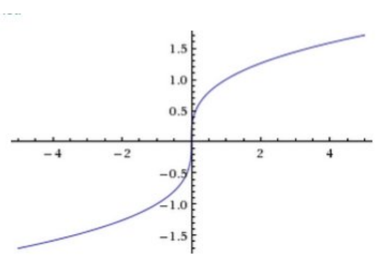


FIGURE 4.0.9. $f(x) = x^{\frac{1}{3}}$

In the next theorem we prove that a differential function is continuous. Although most of our proofs have been relegated to Chapter 12, every once in a way there is a proof that is both simple and important and worth seeing right away.

[7] Study note: Our approach to limits is informal in the sense that we are not using the $\delta - \epsilon$ definition. We end up drawing a figure to compute limits as in Figure 4.0.8. So we may as well recognize directly from the graph of the function itself (Figure 4.0.9) when a derivative does not exist. The graph gives a lot of information provided we read it correctly. This visual approach is important for conceptual reasons as we go along, not just for solving one or two problems.

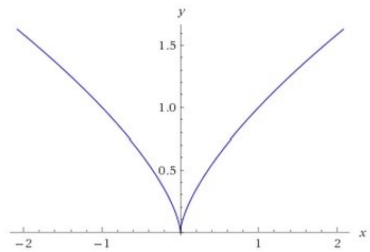


FIGURE 4.0.10. $f(x) = \sqrt[3]{x^2}$

Theorem 4.5. If a function f is differentiable at a, then f is continuous at a.

Proof. To prove that f is continuous at a, we must show that $\lim_{x \to a} f(x) = f(a)$. Observe that we can write $f(x)$ algebraically as

$$f(x) = \frac{f(x) - f(a)}{x - a}(x - a) + f(a)$$

To see this simplify the right side of the above expression and we get precisely $f(x)$. So

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a) + f(a) \\
&= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a}(x - a) + \lim_{x \to a} f(a) \\
&= f'(a)(a - a) + f(a) \\
&= f(a)
\end{aligned}$$

□

The converse of Theorem 4.5 is not true. We already saw that $f(x) = |x|$ is a function that is continuous everywhere, but not differentiable at 0. The general thinking among mathematicians of the 18th century was that a continuous function would have just a few non-differentiable points. Then German mathematician Karl Weierstrass (1815 - 1897) found a function that was continuous everywhere and differentiable nowhere. The function is given in the format of a power series (which we will study later). It is given by

$$f(x) = \sum_{k=1}^{\infty} a^k \cos b^k \pi x$$

where $0 < a < 1$ and b is any odd interger such that $ab > \frac{3\pi}{2} + 1$. A specific example of such a function where $a = \frac{1}{2}$, $b = 3$ and $n = 3$ is given below and its graph is shown in Figure 4.0.11. [8] The graph on the left of Figure 4.0.11 is a zoomed in version of the graph on the

[8]Enter this function as `f(x) = cos(3*x*pi)/2 + cos(9*x*pi)/4 + cos(27*x*pi)/8`.

right.

$$f(x) = \frac{1}{2}\cos 3\pi x + \frac{1}{4}\cos 9\pi x + \frac{1}{8}\cos 27\pi x$$

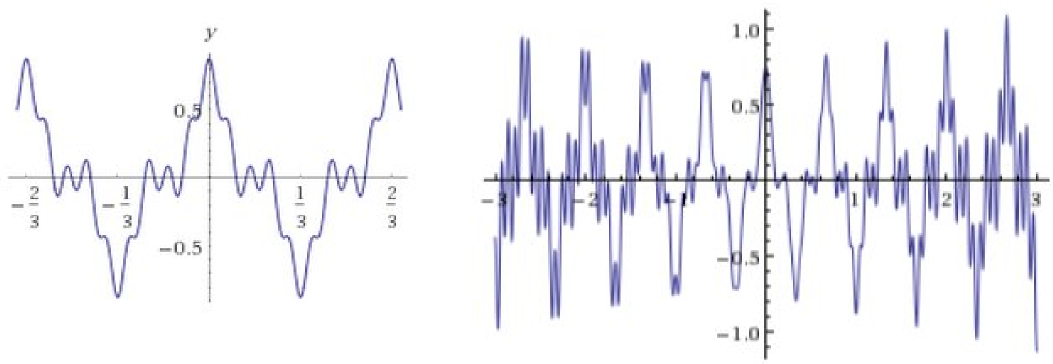


FIGURE 4.0.11. $f(x) = \frac{1}{2}\cos 3\pi x + \frac{1}{4}\cos 9\pi x + \frac{1}{8}\cos 27\pi x$

Such anomalous functions troubled the mathematical literati of the 18th and 19th centuries. French mathematician Henri Poincaré (1854 - 1912) called these functions **monsters** and declared Weierstrass's work "an outrage against common sense" and Charles Hermite (1822 - 1901) said (jokingly we presume),

> I turn with terror and horror from this lamentable scourge of functions with no derivatives.

If we don't stop n at 3 as in the previous example, and instead let it get larger and larger tending toward infinity, Weietstrass' Monster has the fractal appearance shown in Figure 4.0.12. [9] Anywhere you zoom in the function looks the same. Weietstrass's work laid the foundations of the modern theory of limits. Notice how much later this is compared to Newton and Leibniz's original discovery.

Practice Problems

(1) Use the formal definition of the derivative to find $f'(x)$.

a) $f(x) = 3x^2$

b) $f(x) = x^4$

c) $f(x) = 6$

[9] Image taken from `http://commons.wikimedia.org/wiki/File:WeierstrassFunction.svg`. `An animated gif showing the fractal nature of the Monster may be found at http://www.math.washington.edu/ conroy/general/weierstrass/weier01.gif.`

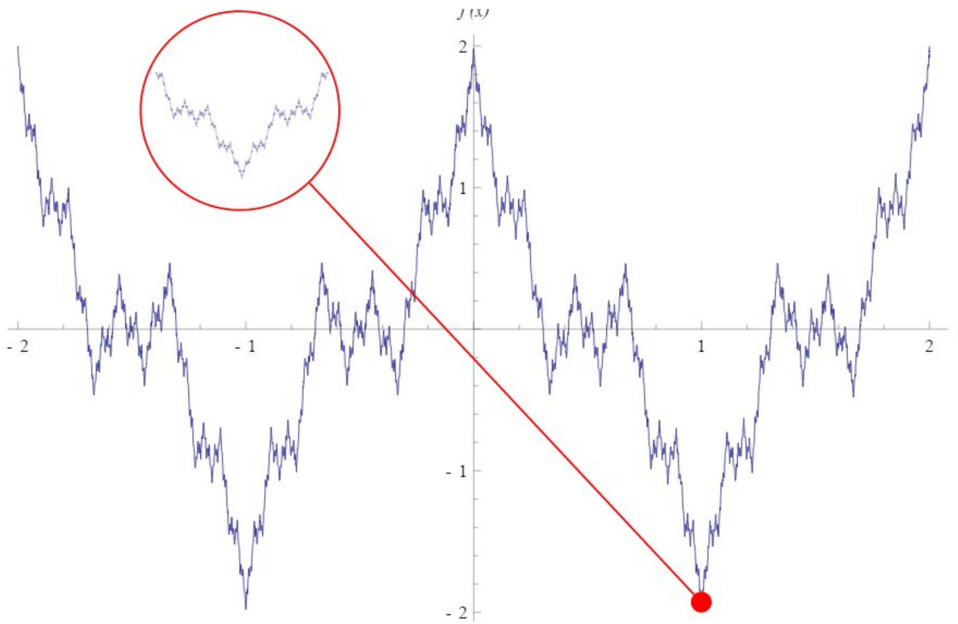


FIGURE 4.0.12. Weietstrass' Monster

d) $f(x) = c$, where c is a constant
e) $f(x) = x$
f) $f(x) = \sqrt{x+2}$
g) $f(x) = \frac{1}{x}$
h) $f(x) = \frac{1}{x^2}$
i) $f(x) = \frac{x}{x+1}$
j) $f(x) = \frac{1}{3x-1}$
k) $f(x) = \frac{1}{2x+1}$
l) $f(x) = \frac{4}{2-3x}$
m) $f(x) = \sqrt{3-6x}$

(2) Determine where the function is not differentiable. Explain your answer by drawing a graph and writing a sentence or two.

a) $f(x) = |x|$
b) $f(x) = \sqrt{x}$
c) $f(x) = \sqrt[3]{x}$
d) $f(x) = |x-2| + 3$
e) $f(x) = \sqrt{x+5}$
f) $f(x) = \sqrt{4-x^2}$

CHAPTER 5

Derivative Rules and Formulas

Before we embark on two chapters filled with techniques for finding the derivative let us hear the story of the infamous rivalry between Newton and Leibniz and the reaction of people to mathematics and science. The backdrop to the development of Calculus is 17th and 18th century Europe. Mathematicians and scientists were viewed with suspicion. Poet and painter William Blake (1785 - 1895), a deeply religious man opposed to science, painted a picture of Newton (see Figure 5.0.1) holding a compass - an instrument perceived as clipping the wings of imagination. The page on which Newton is drawing flows out of his head. [1]

FIGURE 5.0.1. William Blake's 1795 painting of Newton

Blake was a leading figure in the Romantic era that arose in opposition to the Age of Enlightenment that represented reason. Blake believed

> Art is the tree of life.
>
> Science is the tree of death

Perhaps the artisitic and creative aspect of mathematics was not fully understood back then. Poet John Keats wrote

> Philosophy will clip an Angel's wings
>
> Conquer all mysteries by rule and line,
>
> Empty the haunted air, and gnomed mine -
>
> Unweave a rainbow ...

[1] https://blogs.stsci.edu/livio/2014/10/22/on-william-blakes-newton/

and influenced Edger Allen Poe to write his famous sonnet “To Science,” [2]

> Science! true daughter of Old Time thou art!
>
> Who alterest all things with thy peering eyes.
>
> Why preyest thou thus upon the poet’s heart,
>
> Vulture, whose wings are dull realities?

One cannot say this era has completely dissapeared. An article in the Guardian titled “What makes some people so suspicious of the findings of science?” provides a straightforward answer. [3]

> The scientific method leads us to truths that are less than self-evident, often mind-blowing and sometimes hard to swallow. In the early 17th century, when Galileo claimed that the Earth spins on its axis and orbits the sun, he wasn’t just rejecting church doctrine. He was asking people to believe something that defied common sense - because it sure looks like the sun’s going around the Earth, and you can’t feel the Earth spinning. Galileo was put on trial and forced to recant. Two centuries later, Charles Darwin escaped that fate. But his idea that all life on Earth evolved from a primordial ancestor and that we humans are distant cousins of apes, whales and even deep-sea molluscs is still a big ask for a lot of people.

In his lifetime, Newton got into a bitter rivalry with Leibniz that lasted till his death. This was somewhat unusual because Newton was also quite humble judging by his most famous quote, “If I have seen further than others, it is by standing upon the shoulders of giants.” He died feeling robbed of his greatest achievement. In this he appears to have been aided by well-meaning but trouble-making friends.

> In 1695 Newton received a letter from his Oxford mathematician friend John Wallis, containing news that cast a cloud over the rest of his life. Writing about Newton’s early mathematical discoveries, Wallis warned him that in Holland “your notions” are known as “Leibniz’s Calculus Differentialis,” and he urged Newton to take steps to protect his reputation. At that time the relations between Newton and Leibniz were still cordial and mutually respectful. However, Wallis’s letters soon curdled the atmosphere, and initiated the most prolonged, bitter, and damaging of all scientific quarrels: the famous (or infamous) Newton-Leibniz priority controversy over the invention of calculus.
>
> It is now well established that each man developed his own form of calculus independently of the other, that Newton was first by 8 or 10 years but did not publish his ideas, and that Leibniz’s papes of 1684 and 1686 were the earliest publications on the subject. However, what are now perceived as simple facts were not nearly so clear at the time. There were ominous minor rumblings for years after Wallis’ letters, as the storm gathered.

[2] These selections should in no way diminish the greatness of these poets in non-scientific matters.

[3] “`http://www.theguardian.com/science/2015/feb/27/science-facts-findings-method-scepticism.`

> What began as mindless innuendoes rapidly escalated into blunt charges of plagiarism on both sides. Egged on by followers anxious to win a reputation under his auspices, Newton allowed himself to be drawn into the centre of the fray and once his temper was aroused by accusations of dishonesty, his anger was beyond constraint. Leibniz's conduct of the controversy was not pleasant, and yet it paled beside that of Newton. Although he never appeared in public, Newton wrote most of the pieces that appeared in his defense, publishing them under the names of his yound men, who never demurred. As president of Royal Society, he appointed an "impartial" committee to investigate the issue, secretly wrote the report officially published by the society [in 1712], and reviewed it anonymously in the *Philosophical Transactions*. Even Leibniz's death could not allay Newton's wrath, and he continued to pursue the enemy beyond the grave. The battle with Leibniz, the irrepressible need to efface the charge of dishonesty, dominated the final 25 years of Newton's life. Almost any paper on any subject from those years is apt to be interrupted by a furious paragraph against the German philosopher as he honed the instruments of his fury ever more keenly (Richard S. Westfall, in *Encyclopaedia Britannica*).

All this was bad enough, but the disastrous effect of the controversy on British science and mathematics was much more serious. It became a matter of patriotic loyalty for the British to use Newton's geometrical methods and clumsy calculus notations, and to look down their noses at the upstart work being done on the Continent. However, Leibniz's analytical methods proved to be far more fruitful and effective, and it was his followers who were the moving spirits in the richest period of development in mathematical history. What has been called "the Great Sulk" continued; for the British, the work of the Bernoullis, Euler, Lagrange, Laplace, Gauss, and Riemann remained a closed book; and British mathematicians sank into a coma of impotence and irrelevancy that lasted through most of the eighteenth and nineteeth centuries.

Theorem 5.1. (The Formulas)

(1) Let $f(x) = c$ where c is a real number, then $f'(x) = 0$

(2) Let $f(x) = ax + b$ where a and b are real numbers, then $f'(x) = a$

(3) **(Power Rule)** Let $f(x) = x^n$ where n is a real number, then $f'(x) = nx^{n-1}$

(4) Let $f(x) = e^x$, then $f'(x) = e^x$

(5) Let $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$

(6) Let $f(x) = b^x$, where b is a real number, then $f'(x) = b^x \ln b$

(7) Let $f(x) = \log_b x$ where b is a real number, then $f'(x) = \frac{1}{x \ln b}$

Theorem 5.2. The Rules

(1) **(Sum Rule)** If $k(x) = f(x) + g(x)$, then $k'(x) = f'(x) + g'(x)$

(2) **(Product Rule)** If $k(x) = f(x)g(x)$, then $k'(x) = f(x)g'(x) + g(x)f'(x)$

(3) **(Quotient Rule)** If $k(x) = \frac{f(x)}{g(x)}$, then $k'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

(4) **(Chain Rule)** If $k(x) = (f \circ g)(x)$, then $k'(x) = f'(g(x))g'(x)$

We will prove all of these rules and formulas except for a portion of Formula (3), Formula (4), and Rule (4) because their proofs are slightly more complicated. We save them for Chapter 12. [4]

Proofs of Formulas (1) (2) and Part of (3)

(1) Let $f(x) = c$. Then, by definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$$

(2) Let $f(x) = ax + b$. Then, by definition

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{a(x+h) + b - (ax+b)}{h} \\ &= \lim_{h \to 0} \frac{ax + ah + b - ax - b}{h} \\ &= \lim_{h \to 0} \frac{ah}{h} \\ &= a \end{aligned}$$

(3) Let $f(x) = x^n$. We will break this proof into three parts: suppose n is a natural number; suppose n is a negative natural number; and suppose n is a real number.

The proof for each sub-part is different. We will prove the first part and save the other two for later. Observe that if $n = 0$, then we get $f(x) = x^0 = 1$, so $f'(x) = 0$. The formula still holds since $nx^{n-1} = 0x^{0-1} = 0$.

Proof of Formula (3a): Suppose $f(x) = x^n$, where n is a natural number. We will use the Binomial Theorem. Let n be a natural number and a, b be real numbers.

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots \binom{n}{n-1} a b^{n-1} + b^n$$

[4] A student taking "Survey of Calculus" may omit all proofs and go straight to Examples 5.1 and 5.2.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{x\to 0}\frac{(x+h)^n - x^n}{h}\\
&= \lim_{h\to 0}\frac{x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots h^n - x^n}{h}\\
&= \lim_{h\to 0}\frac{\binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots h^n}{h}\\
&= \lim_{h\to 0}\frac{h[\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots h^{n-1}]}{h}\\
&= \lim_{h\to 0}\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots h^{n-1}\\
&= \binom{n}{1}x^{n-1}\\
&= nx^{n-1}
\end{aligned}$$

Proof of Rules (1) (2) and (3)

(1) Let $k(x) = f(x) + g(x)$.

$$\begin{aligned}
k'(x) &= \lim_{h\to 0}\frac{k(x+h)-k(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-[f(x)+g(x)]}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x) + g'(x)
\end{aligned}$$

(2) Let $k(x) = f(x)g(x)$.

$$\begin{aligned}
k'(x) &= \lim_{h\to 0}\frac{k(x+h)-k(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\\
&= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

(3) Let $k(x) = \frac{f(x)}{g(x)}$.

$$\begin{aligned}
k'(x) &= \lim_{h\to 0} \frac{k(x+h) - k(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0} \frac{g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}}{g(x+h)g(x)} \\
&= \frac{\lim_{h\to 0} g(x)\frac{f(x+h)-f(x)}{h} - \lim_{h\to 0} f(x)\frac{g(x+h)-g(x)}{h}}{\lim_{h\to 0} g(x+h)g(x)} \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}
\end{aligned}$$

Note that in the third line of Proof (2) we add and subtract $f(x+h)g(x)$. This clever and perfectly legitimate algebraic trick makes the proof work. Proof (3) also has such a trick. We add and subtract $f(x)g(x)$ in the third line.

There is one more piece of notation to introduce and that is the $\frac{dy}{dx}$ notation. The expression dy represents a small change in y and dx represents a small change in x. Put together $\frac{dy}{dx}$ represents the slope of a small secant line, and as we know, when the interval around a point shrinks, the secant line becomes the tangent line at the point. So as the secant line tends toward the tangent line, the slope of the secant line tends toward the slope of the tangent line. This notation is much better than the $f'(x)$ notation when using the rules and is a good example of how the right kind of notation makes things easier.

Example 5.3. Find the derivatives for the following functions:

(1) $y = x^8 + 12x^5 + 10x^3 - 6x + 5$

(2) $y = \sqrt{x}$

(3) $y = \frac{x^2+2}{x^3-5x}$

(4) $y = (x^2 + 2x - 5)^{10}$

(5) $y = \sqrt{3x + 1}$

(6) $y = \sqrt{x}(2x + 3)^5$

Solutions.

(1) $y = x^8 + 12x^5 + 10x^3 - 6x + 5$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^8 + 12x^5 + 10x^3 - 6x + 5)\\ &= \frac{d}{dx}x^8 + \frac{d}{dx}12x^5 + \frac{d}{dx}10x^3 - \frac{d}{dx}6x + \frac{d}{dx}5 \quad \textbf{Sum Rule}\\ &= \frac{d}{dx}x^8 + 12\frac{d}{dx}x^5 + 10\frac{d}{dx}x^3 - 6\frac{d}{dx}x + \frac{d}{dx}5 \quad \textbf{Formula 2}\\ &= 8x^7 + 12(5x^4) + 10(3x^2) - 6 + 0 \quad \textbf{Formulas 1 and 3}\\ &= 8x^7 + 60x^4 + 30x^2 - 6\end{aligned}$$

Note: The Sum Rule and Formulas 1, 2, and 3 are used in this problem. This, being the first example, we wrote all the steps. In practice, all of the steps are done in the head. [5] We write directly

$$\frac{dy}{dx} = 8x^7 + 60x^4 + 30x^2 - 6$$

(2) $y = \sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \quad \textbf{Formula 1}$$

(3) $y = \frac{x^2+2}{x^3-5x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^3-5x)\frac{d}{dx}(x^2+2) - (x^2+2)\frac{d}{dx}(x^3-5x)}{(x^3-5)^2} \quad \textbf{Quotient Rule}\\ &= \frac{(x^3-5x)(2x) - (x^2+2)(3x^2-5)}{(x^3-5)^2}\end{aligned}$$

(4) $y = (x^2 + 2x - 5)^{10}$

$$\begin{aligned}\frac{dy}{dx} &= 10(x^2+2x-5)^9\frac{d}{dx}(x^2+2x-5) \quad \textbf{Formula 1 and Chain Rule}\\ &= 10(x^2+2x-5)^9(2x+2)\end{aligned}$$

[5] There is an art to which steps must be written and which may be omitted. Everyone agrees that skipping too many steps leads to mistakes, but writing too much also makes things difficult and boring. There is no reason to show implementation of the rules separately step by step.

(5) $y = \sqrt{3x+1}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}(3x+1)^{-\frac{1}{2}}\frac{d}{dx}(3x+1) \quad \textbf{Formula 1 and Chain Rule} \\ &= \frac{1}{2\sqrt{3x+1}}(3) \\ &= \frac{3}{2\sqrt{3x+1}} \end{aligned}$$

(6) $y = \sqrt{x}(2x+3)^5$

$$\begin{aligned} \frac{dy}{dx} &= \sqrt{x}\frac{d}{dx}(2x+3)^5 + (2x+3)^5\frac{d}{dx}\sqrt{x} \quad \textbf{Product Rule} \\ \frac{dy}{dx} &= \sqrt{x}[5(2x+3)^4]\frac{d}{dx}(2x+3) + (2x+3)^5\frac{1}{2}(x^{-\frac{1}{2}}) \quad \textbf{Formula 3 and Chain Rule} \\ &= \sqrt{x}[5(2x+3)^4]2 + (2x+3)^5\frac{1}{2\sqrt{x}} \\ &= 10\sqrt{x}(2x+3)^4 + \frac{(2x+3)^5}{2\sqrt{x}} \end{aligned}$$

□

With the Quotient Rule we can easily prove the second part of Formula (3), when n is a negative integer.

Proof of Formula (3b) Suppose $f(x) = x^{-n}$, where n is a natural number. Then $f(x) = \frac{1}{x^n}$. So we can apply the Quotient Rule.

$$\begin{aligned} f'(x) &= \frac{x^n \frac{d}{dx}(1) - (1)\frac{d}{dx}(x^n)}{(x^n)^2} \\ &= \frac{x^n(0) - (1)nx^{n-1})}{(x^n)^2} \\ &= \frac{-nx^{n-1}}{x^{2n}} \\ &= \frac{-nx^{n-1}}{x^{2n}} \\ &= -n(x^{n-1-2n}) \\ &= -nx^{-n-1} \end{aligned}$$

□

Returning to the formulas, the proof that the derivative of e^x is itself is more complicated than one would expect. We give an informal proof here and postpone the formal proof until Chapter 12. [6]

Informal Proof of Rule (4)

Let $f(x) = e^x$. Then,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h\to 0} \frac{e^x e^h - e^x}{h} \\ &= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} \\ &= \lim_{h\to 0} e^x \lim_{h\to 0} \frac{e^h - 1}{h} \\ &= e^x \lim_{h\to 0} \frac{e^h - 1}{h} \end{aligned}$$

Observe from Figure 5.0.2 that $\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1$. So

$$f'(x) = e^x$$

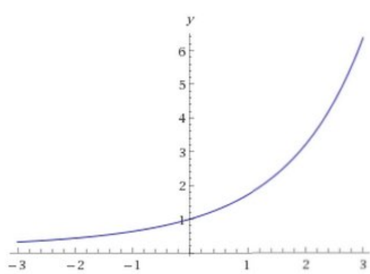


FIGURE 5.0.2. $y = \frac{e^x-1}{x}$

The fact that the derivative of e^x is e^x was discovered by Jacob Bernoulli in 1683. This is quite an amazing fact indeed! French chemical engineer and writer Francois le Lionnais (1901 - 1984) [7] describes how awe-inspiring it is as follows:

[6]There is considerable value in informal proofs as it gives the proof idea. Knowing the difference between a proof idea and a formal proof is an important skill that comes only after seeing many proofs. Informal proofs are fine as long as one is aware they are informal.

[7]Francois le Lionnais was a founder of the Oulipo style of writing. A group of French writers and mathematicians got together and devised ways of writing with constraints, often mathematical constraints, that were meant to trigger creativity. For example, writing an essay without a specific letter of the alphabet. His quote appears in *Great Currents of Mathematical Thought*, vol. 1, Dover Publications.

> Who has not been amazed to learn that the function $y = e^x$, like a phoenix rising from its own ashes, is its own derivative?

Look at the graphs $y = 2^x$ and $y = 3^x$ in Figure 5.0.3. Draw the tangent line to both curves at $(0, 1)$ and you will find the slopes to be roughly 0.693 and 1.098. This means for some value of b between 2 and 3, the curve $y = b^x$ must have slope of the tangent line at $(0, 1)$ exactly equal to 1. The value of b happens to be $2.71828\ldots$. Later, in 1737 Leonard Euler proved that this number is irrational and we call it e in his honor.

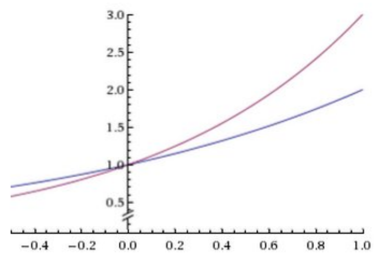


FIGURE 5.0.3. $y = 2^x$ and $y = 3^x$

Proofs of Formulas (5), (6), (7)

(5) Let $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$

Proof. Since log and exponential functions are inverses of each other,

$$e^{\ln x} = x$$

Taking derivatives on both sides

$$\frac{d}{dx} e^{\ln x} = \frac{d}{dx} x$$

Using Formula (4) and the Chain Rule

$$e^{\ln x} \frac{d}{dx} \ln x = 1$$

$$x \frac{d}{dx} \ln x = 1$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

(6) Let $f(x) = b^x$, where b is a real number, then $f'(x) = (b^x) \ln b$

Proof.

$$b^x = e^{\ln b^x}$$

Using log properties

$$b^x = e^{x \ln b}$$

Taking derivatives on both sides and using Formula (4) and the Chain Rule

$$\begin{aligned}\frac{d}{dx}b^x &= \frac{d}{dx}e^{x\ln b}\\ &= e^{x\ln b}\frac{d}{dx}x\ln b\\ &= e^{x\ln b}(lnb)\\ &= e^{\ln b^x}(lnb)\\ &= b^x(lnb)\end{aligned}$$

(7) Let $f(x) = \log_b x$ where b is a real number, then $f'(x) = \frac{1}{x\ln b}$

Proof.

$$b^{\log_b x} = x$$

Taking derivatives on both sides

$$\frac{d}{dx}b^{\log_b x} = \frac{d}{dx}x$$

Using Formula (6) and the Chain Rule

$$b^{\log_b x}\ln b(\frac{d}{dx}\log_b x) = 1$$

$$x\ln b(\frac{d}{dx}\log_b x) = 1$$

$$\frac{d}{dx}\log_b x = \frac{1}{x\ln b}$$

□

Next, we give a short alternate proof of Formula (7) using the Change of Base formula for logs. If we know $\log_b x$ then we can find $\log_a x$ for any desired base a as follows:

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

Alternate Proof of Formula (7). We have to find the derivative of $f(x) = \log_b x$. Using the Change of Base formula, we can change the base to e as follows:

$$f(x) = \log_b x = \frac{\ln x}{\ln b} = (\frac{1}{\ln b})\ln x$$

Using Formula (5) and the fact that $\ln b$ is a constant

$$f'(x) = \frac{1}{x\ln b}.$$

□

That's it! As promised it is a short proof.

EXAMPLE 5.4. Find the derivatives of the following functions:

(1) $y = xe^x$

(2) $y = x^2e^x$

(3) $y = x^2b^x$

(4) $y = (x^2 + 2x + 1)\ln x$

(5) $y = \frac{\log_b x}{x^2+1}$

Solutions.

(1) $y = xe^x$

$$\begin{aligned}\frac{dy}{dx} &= x\frac{d}{dx}e^x + e^x\frac{d}{dx}x \\ &= xe^x + e^x = e^x(x+1)\end{aligned}$$

(2) $y = x^2e^x$

$$\begin{aligned}\frac{dy}{dx} &= x^2\frac{d}{dx}e^x + e^x\frac{d}{dx}x^2 \\ &= x^2e^x + e^x(2x) = xe^x(x+2)\end{aligned}$$

(3) $y = x^2b^x$

$$\begin{aligned}\frac{dy}{dx} &= x^2\frac{d}{dx}b^x + b^x\frac{d}{dx}x^2 \\ &= x^2b^x\ln b + b^x(2x) = xb^x(x\ln b + 2)\end{aligned}$$

(4) $y = (x^2 + 2x + 1)\ln x$

$$\begin{aligned}\frac{dy}{dx} &= (x^2 + 2x + 1)\frac{d}{dx}\ln x + (\ln x)\frac{d}{dx}(x^2 + 2x + 1) \\ &= (x^2 + 2x + 1)\frac{1}{x} + (\ln x)(2x + 2)\end{aligned}$$

(5) $y = \frac{\log_b x}{x^2+1}$

$$
\begin{aligned}
&= \frac{d}{dx} \frac{(x^2+1)\frac{d}{dx}\log_b x - (\log_b x)\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\
&= \frac{(x^2+1)\frac{1}{x\ln b} - (\log_b x)(2x)}{(x^2+1)^2}
\end{aligned}
$$

□

The second derivative of a function is the derivative of the derivative. The third derivative is the derivative taken three times, and so on. The nth derivative is denoted as $\frac{d^2y}{dx^2}$. These derivatives are called **higher order derivatives**.

Example 5.5. Find the second derivative of the following functions:

(1) $y = x^3 + 5x^2 - 2x - 1$

(2) $y = xe^x$

Solutions.

(1) $y = x^3 + 5x^2 - 2x - 1$

$$
\begin{aligned}
\frac{dy}{dx} &= 3x^2 + 10x - 2 \\
\frac{d^2y}{dx^2} &= 6x + 10
\end{aligned}
$$

(2) $y = xe^x$

$$
\frac{dy}{dx} = xe^x + e^x = e^x(x+1)
$$

$$
\begin{aligned}
\frac{d^2x}{dy^2} &= e^x\frac{d}{dx}(x+1) + (x+1)\frac{d}{dx}e^x \\
&= e^x + (x+1)e^x \\
&= e^x(x+2)
\end{aligned}
$$

Former President of the American Mathematical Society Hugo Rossi noted in a column in the AMS *Notices* Vol. 43 No. 10:

> In the fall of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting president used the third derivative to advance his case for reelection...Was President Nixon

telling us that the economy was getting better? Did his listeners understand that in fact the inflation rate was still increasing and thus the economy still worsening?

Inflation is defined as a sustained increase in the general price level of goods and services in an economy over a period of time. So if x represents goods and services and y their prices, inflation is the rate of change in price. In other words inflation is the first derivative. The rate of change of inflation is the second derivative, and the rate of change of the rate of change of inflation is the third derivative.

Practice Problems

(1) Differentiate using the sum, difference, product and quotient rules.

a) $y = x^4 + 3x^2 - 2$
b) $y = (1 + 4x)(x^2 + x)$
c) $y = \frac{x^2+1}{x^2-1}$
d) $y = \frac{x^2+4x+3}{\sqrt{x}}$
e) $y = \frac{x^3}{1-x^2}$
f) $y = \frac{x^2+2}{x^4-3x^2+1}$

(2) Differentiate. In addition to the above rules, the Chain Rule is needed for these problems.

a) $y = (x^4 + 3x^2 - 2)^2$
b) $y = (4x - x^2)^{100}$
c) $y = \sqrt[4]{x^3 + 2x + 1}$
d) $y = (x^4 + 1)^{\frac{2}{3}}$
e) $y = \frac{1}{(x^4+1)^3}$
f) $y = (1 + 4x)^5(3 + x - x^2)^8$
g) $y = \left(\frac{x^2+1}{x^2-1}\right)^3$
h) $y = \sqrt{\frac{x-1}{x+1}}$

(3) Differentiate.

a) $y = e^{2x}$
b) $y = e^{1-x}$

c) $y = e^{x^2-2x}$

d) $y = e^{-x^2}$

e) $y = e^{-\frac{3}{x^2}}$

f) $y = xe^x$

g) $y = \frac{e^x}{x+1}$

h) $y = (e^x + e^{-x})^3$

i) $y = \frac{e^x - e^{-x}}{2}$

j) $y = x^2e^x$

k) $y = x^2e^x - 2xe^x + 2e^x$

l) $y = 3^{2x}$

m) $y = 2^{1-x}$

n) $y = 5^{x^2-2x}$

o) $y = 2^{-x^2}$

p) $y = 2^{\frac{-3}{x^2}}$

q) $y = x3^x$

r) $y = \frac{2^x}{x+1}$

(4) Differentiate.

a) $y = \ln 2x$

b) $y = \log_{10} 2x$

c) $y = \log_{10} x^3$

d) $y = \ln(x^2 + 5x)$

e) $y = \log_3(x^2 + 5x)$

f) $y = \frac{\ln x}{\ln 2x+1}$

g) $y = \frac{\ln x}{x^2}$

h) $y = \frac{1+\ln x}{1-\ln x}$

i) $y = \frac{1}{\ln x}$

(5) Find the equation of the tangent line to the curve at the indicated point.

a) $y = x^2$ at $(3, 9)$

b) $y = \frac{1}{x}$ at $(1, 1)$

c) $y = x^{\frac{5}{2}}$ at $(3, 3^{\frac{5}{2}})$

(6) The equation of motion of a particle is given by $y = 40t - 16t^2$ where y is in meters and t is in seconds. Velocity is the rate of change of distance (it is speed with a direction). Acceleration is the rate of change of velocity.

a) Find the velocity as a function of t.
b) Find the velocity when $t = 2$.
c) Find the acceleration as a function of t.
d) Find the acceleration when $t = 2$.

(7) Suppose a rock is thrown straight up in the air with an initial velocity of 40 feet per second. Its height after t seconds is given by $y = 40t - 16t^2$.

a) Find the velocity when $t = 2$.
b) At what time will the rock hit the surface?
c) With what velocity will the rock hit the surface?

CHAPTER 6

More Derivative Formulas

This chapter is essentially a continuation of the previous chapter with more formulas. We will study the derivatives of the six trigonometric functions and the six inverse trigonometric functions adding twelve more formulas to the previous list of six formulas and four rules. [1] British author William Somerset Maugham said

> There are three rules for writing a novel.
> Unfortunately, no one knows what they are.

Fortunately, we have many wonderful rules for finding derivatives that make things quite easy. We don't always get so lucky, not in math, not in anything else. So when there are rules that work why not study them for their own sake and find pleasure in the symmetry and beauty of the rules? From now on we will use the $\frac{dy}{dx}$ notation most of the time.

Theorem 6.1. (Derivatives of the six trigonometric functions)

(1) $\frac{d}{dx}\sin x = \cos x$

(2) $\frac{d}{dx}\cos x = -\sin x$

(3) $\frac{d}{dx}\tan x = \sec^2 x$

(4) $\frac{d}{dx}\cot x = -\csc^2 x$

(5) $\frac{d}{dx}\sec x = \sec x \tan x$

(6) $\frac{d}{dx}\csc x = -\csc x \cot x$

We will give an informal proof for (1) since it involves two limits that cannot be found by the techniques in Chapter 3 and formal proofs for the rest. The formal proof of (1) appears in Chapter 12. The proofs of Formulas (2) to (6) use the following trignometic identities:

- $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
- $\tan x = \frac{\sin x}{\cos x}$
- $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$
- $\sec x = \frac{1}{\cos x}$

[1] A student taking a course in Survey of Calculus or Business Calculus should focus on the twelve formulas and the two examples based on them, omitting all proofs.

- $\csc x = \frac{1}{\sin x}$.
- $\sin^2 x + \cos^2 x = 1$
- $\sec^2 x - \tan^2 x = 1$
- $\csc^2 x - \cot^2 x = 1$

But before looking at the proofs let us use these formulas in examples.

Example 6.2. Find the derivatives of the following functions:

(1) $y = x^2 \sin x$

(2) $y = e^{\cos x}$

(3) $y = e^{\sec 3x}$

(4) $y = \frac{\sec x}{1+\tan x}$

Solutions.

(1) $y = x^2 \sin x$

$$\begin{aligned} \frac{dy}{dx} &= x^2 \frac{dy}{dx} \sin x + \sin x \frac{d}{dx} x^2 \\ &= (x^2)(\cos x) + (\sin x)(2x) \end{aligned}$$

(2) $y = e^{\cos x}$

$$\begin{aligned} \frac{dy}{dx} &= e^{\sec 3x} \frac{d}{dx} \sec 3x \\ &= e^{\sec 3x} (\sec 3x \tan 3x) \frac{d}{dx} 3x \\ &= e^{\sec 3x} (\sec 3x \tan 3x)(3) \end{aligned}$$

(3) $y = e^{\sec 3x}$

$$\begin{aligned} \frac{dy}{dx} &= e^{\cos x} \frac{d}{dx} \cos x \\ &= e^{\cos x}(-\sin x) \end{aligned}$$

(4) $y = \frac{\sec x}{1+\tan x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+\tan x)\frac{d}{dx}\sec x - (\sec x)\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}\\ &= \frac{(1+\tan x)\sec x\tan x - (\sec x)\sec^2 x}{(1+\tan x)^2}\\ &= \frac{(1+\tan x)\sec x\tan x - \sec^3 x}{(1+\tan x)^2}\end{aligned}$$

□

Note that it may be possible to simply further using identities and get an elegant final answer, but for the most part, this much simplification is enough. However, if we have to find the second derivative, then more simplication is needed to get an expression easier to differentiate again.

Informal Proof of (1). Using the definition of the derivative

$$\begin{aligned}\frac{d}{dx}\sin x &= \frac{\sin(x+h)-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x\cos h - \sin x + \cos x\sin h}{h}\\ &= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x\sin h}{h}\\ &= \lim_{h\to 0}\sin x\lim_{h\to 0}\frac{\cos h - 1}{h} + \lim_{h\to 0}\cos x\lim_{h\to 0}\frac{\sin h}{h}\end{aligned}$$

We saw the graph of $y = \frac{\sin x}{x}$ in Figure 1.0.14. Recall that $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$. Similarly, the graph of $y = \frac{\cos x - 1}{x}$ is shown in Figure 6.0.1. Observe that $\lim\limits_{x\to 0}\frac{\cos x-1}{x} = 0$. So

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{h\to 0}\sin x\lim_{h\to 0}\frac{\cos h - 1}{h} + \lim_{h\to 0}\cos x\lim_{h\to 0}\frac{\sin h}{h}\\ &= (\sin x)(0) + (\cos x)(1)\\ &= \cos x.\end{aligned}$$

□

Formal Proofs of (2) to (6).

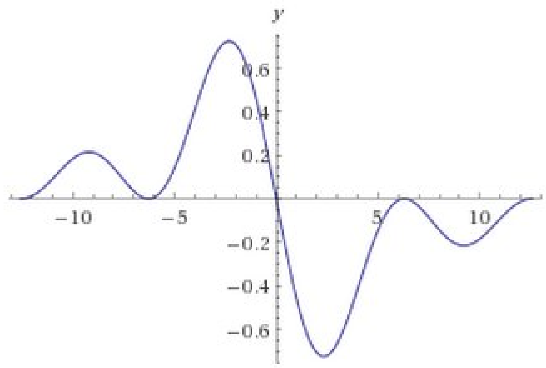


FIGURE 6.0.1. $y = \frac{\cos x - 1}{x}$

(2) $\frac{d}{dx} \cos x = -\sin x$

Using the Chain Rule and trigonometric identity $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

$$\begin{aligned}\frac{d}{dx} \cos x &= \frac{d}{dx} \sin(\frac{\pi}{2} - x) \\ &= \cos(\frac{\pi}{2} - x) \frac{d}{dx} (\frac{\pi}{2} - x) \\ &= (\sin x)(-1) \\ &= -\sin x\end{aligned}$$

(3) $\frac{d}{dx} \tan x = \sec^2 x$

Using the Quotient Rule and trognometric identities $\tan x = \frac{\sin x}{\cos x}$ and $\sin^2 x + \cos^2 x = 1$

$$\begin{aligned}\frac{d}{dx} \tan x &= \frac{d}{dx} \frac{\sin x}{\cos x} \\ &= \frac{\cos x \frac{d}{dx} \sin x - \sin x \frac{d}{dx} \cos x}{\cos^2 x} \\ &= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x\end{aligned}$$

(4) $\frac{d}{dx} \cot x = -\csc^2 x$

Using the Quotient Rule and trogonometric identities $\cot x = \frac{\cos x}{\sin x}$ and $\sin^2 x + \cos^2 x = 1$

$$\begin{aligned}
\frac{d}{dx}\cot x &= \frac{d}{dx}\frac{\cos x}{\sin x} \\
&= \frac{\sin x \frac{d}{dx}\cos x - \cos x \frac{d}{dx}\sin x}{\sin^2 x} \\
&= \frac{\sin x(-\sin x) - \cos x(-\cos x)}{\cos^2 x} \\
&= \frac{-(\sin^2 x + \cos^2 x)}{\cos^2 x} \\
&= \frac{1}{\sin^2 x} \\
&= \csc^2 x
\end{aligned}$$

(5) $\frac{d}{dx}\sec x = \sec x \tan x$

Using the Power Rule and trigonometric identity $\sec x = \frac{1}{\cos x}$

$$\begin{aligned}
\frac{d}{dx}\sec x &= \frac{d}{dx}\frac{1}{cosx} \\
&= \frac{d}{dx}(\cos x)^{-1} \\
&= -(\cos x)^{-2}\frac{d}{dx}\cos x \\
&= -\frac{1}{\cos^2 x}(-\sin x) \\
&= \frac{1}{\cos x}\frac{\sin x}{\cos x} \\
&= \sec x \tan x
\end{aligned}$$

(6) $\frac{d}{dx}\csc x = -\csc x \cot x$

Using the Power Rule and trigonometric identity $\csc x = \frac{1}{\sin x}$

$$\begin{aligned}
\frac{d}{dx}\csc x &= \frac{d}{dx}\frac{1}{sinx} \\
&= \frac{d}{dx}(\sin x)^{-1} \\
&= -(\sin x)^{-2}\frac{d}{dx}\sin x \\
&= -\frac{1}{\sin^2 x}\cos x \\
&= -\frac{1}{\sin x}\frac{\cos x}{\sin x} \\
&= -\csc x \cot x
\end{aligned}$$

□

Theorem 6.3. (Derivatives of the six inverse trigonometric functions)

(1) $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

(2) $\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$

(3) $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$

(4) $\frac{d}{dx}\cot^{-1} x = -\frac{1}{(1+x^2)}$

(5) $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$

(6) $\frac{d}{dx}\csc^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}$

Example 6.4. Find the derivative:

(1) $y = x\tan^{-1} x$

(2) $y = x\tan^{-1}\sqrt{x}$

(3) $y = \frac{1}{\sin^{-1} x}$

Solutions.

(1) $y = x\tan^{-1} x$

$$\begin{aligned}\frac{dy}{dx} &= x\frac{d}{dx}\tan^{-1} x + \tan^{-1} x\frac{d}{dx}x \\ &= x\frac{1}{1+x^2} + (\tan^{-1} x)(1) = \frac{x}{1+x^2} + \tan^{-1} x\end{aligned}$$

(2) $y = x\tan^{-1}\sqrt{x}$

$$\begin{aligned}\frac{dy}{dx} &= x\frac{d}{dx}\tan^{-1}\sqrt{x} + \tan^{-1}\sqrt{x}\frac{d}{dx}x \\ &= x\frac{1}{1+x^2}\frac{d}{dx}\sqrt{x} + \tan^{-1}\sqrt{x} \\ &= \frac{x}{1+x^2}\frac{1}{\sqrt{x}} + \tan^{-1}\sqrt{x}\end{aligned}$$

(3) $y = \frac{1}{\sin^{-1} x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\sin^{-1} x)^{-1} \\ &= -(\sin^{-1} x)^{-2}\frac{d}{dx}\sin^{-1} x \\ &= -(\sin^{-1} x)^{-2}\frac{1}{\sqrt{1-x^2}}\end{aligned}$$

□

We will give informal proofs for the derivatives of the inverse trigonometric functions. The formal proofs are in Chapter 12. The informal proofs skip over some important details about the domains and ranges of the inverse trigonometric functions and the appearance of $|x|$ in (6) and (7). The formal proofs are in Chapter 12. Arguably one must see the informal proofs before seeing the formal proofs. So there is good reason to do these informal proofs first. Also the strategy presented in the informal proof is called implicit derivative. It is quite an important strategy and the next chapter is entirely devoted to it.

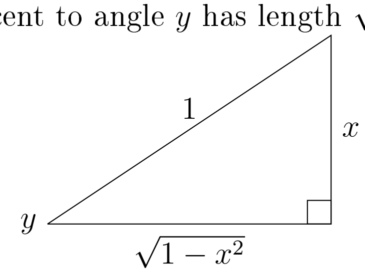
Informal Proofs

(1) $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

Let $y = \sin^{-1} x$. Then $\sin y = x$. Taking derivatives on both sides

$$\begin{aligned}\cos y\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y}\end{aligned}$$

Since $\sin y = x$, we can draw a right triangle with one acute angle labeled y. The side opposite angle y has length x and the hypotenuse has length 1. By the Pythagorean Theorem the side adjacent to angle y has length $\sqrt{1-x^2}$. [2]



From the triangle we see that $\cos y = \frac{\sqrt{1-x^2}}{1}$. So

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

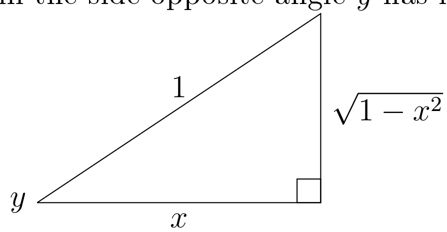
[2] Note that drawing a right triangle may give the mistaken impression that y is the measure of an angle in degrees, but y is in radians, which are real numbers. Limit rules apply to real numbers.

(2) $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$

Let $y = \cos^{-1} x$. Then $\cos y = x$. Taking derivatives on both sides

$$-\sin y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

Since $\cos y = x$, we can draw a right triangle with one acute angle labeled y. The side adjacent to angle y has length x and the hypotenuse has length 1. By the Pythagorean Theorem the side opposite angle y has length $\sqrt{1-x^2}$.



From the triangle we see that $\cos y = \frac{\sqrt{1-x^2}}{1}$. So

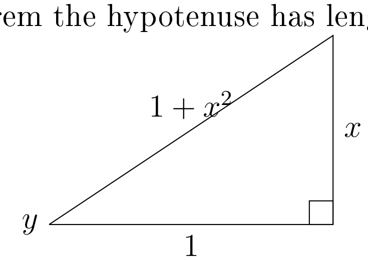
$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

(3) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

Let $y = \tan^{-1} x$. Then $\tan y = x$. Taking derivatives on both sides

$$\sec^2 y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

Since $\tan y = x$, we can draw a right triangle with one acute angle labeled y. The side opposite angle y has length x and the side adjacent to angle y has length 1. By the Pythagorean Theorem the hypotenuse has length $1 + x^2$.



From the triangle we see that $\sec y = \frac{1+x^2}{1}$. So

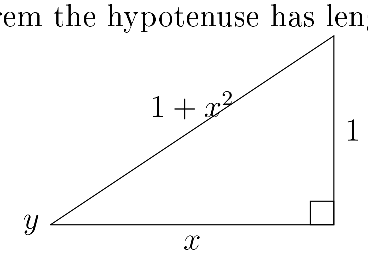
$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+x^2}$$

(4) $\frac{d}{dx}\cot^{-1}x = -\frac{1}{(1+x^2)}$

Let $y = \cot^{-1}x$. Then $\cot y = x$. Taking derivatives on both sides

$$-\csc^2 y\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = -\frac{1}{\csc^2 y}$$

Since $\cot y = x$, we can draw a right triangle with one acute angle labeled y. The side adjacent to angle y has length x and the side opposite angle y has length 1. By the Pythagorean Theorem the hypotenuse has length $1 + x^2$.



From the triangle we see that $\csc y = \frac{1+x^2}{1}$. So

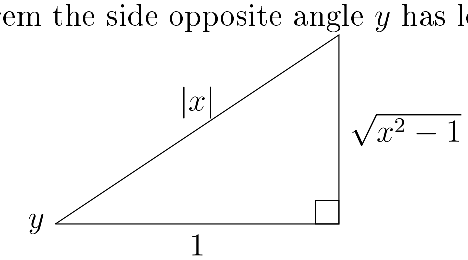
$$\frac{dy}{dx} = -\frac{1}{\csc^2 y} = -\frac{1}{1+x^2}$$

(5) $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$

Let $y = \sec^{-1}x$. Then $\sec y = |x|$. [3] Taking derivatives on both sides

$$\sec y \tan y\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

Since $\sec y = |x|$, we can draw a right triangle with one acute angle labeled y. The side adjacent to angle y has length 1 and the hypotenuse has length $|x|$. By the Pythagorean Theorem the side opposite angle y has length $\sqrt{x^2-1}$.



From the triangle we see that $\sec y = \frac{|x|}{1}$ and $\tan y = \frac{\sqrt{x^2-1}}{1}$. So

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{|x|\sqrt{x^2-1}}$$

[3] To see why we have $|x|$ here were must take a took at the formal proof in Appendix D. Without the formal proof this must be simply believed.

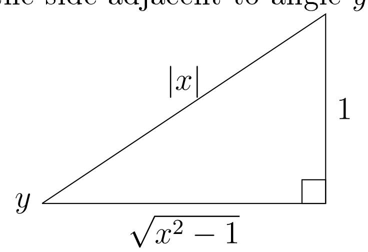
(6) $\frac{d}{dx}\csc^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}$

Let $y = \csc^{-1} x$. Then $\csc y = |x|$.

Taking derivatives on both sides

$$-\csc y \cot y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = -\frac{1}{\csc y \cot y}$$

Since $\csc y = |x|$, we can draw a right triangle with one acute angle labeled y. The side opposite to angle y has length 1 and the hypotenuse has length $|x|$. By the Pythagorean Theorem the side adjacent to angle y has length $\sqrt{x^2-1}$.



From the triangle we see that $\csc y = \frac{|x|}{1}$ and $\cot y = \frac{\sqrt{x^2-1}}{1}$. So

$$\frac{dy}{dx} = -\frac{1}{\csc y \cot y} = -\frac{1}{|x|\sqrt{x^2-1}}$$

□

Practice Problems

(1) Find the derivative:

a) $y = \cos x$
b) $y = \tan x$
c) $y = \csc x$
d) $y = \sec x$
e) $y = \cot x$
f) $y = \sin^{-1} x$
g) $y = \cos^{-1} x$
h) $y = \tan^{-1} x$

(2) Find the derivative:

a) $y = 3x^3 - 2\cos x$
b) $y = x^2 \sin x$
c) $y = \sin x + \frac{1}{2}\cot x$
d) $y = 2\csc x + 5\cos x$
e) $y = x^3 \cos x$
f) $y = 4\sec x + \tan x$
g) $y = \csc x + e^x \cot x$
h) $y = e^x(\cos x + x)$
i) $y = \frac{x}{\tan x + 2}$
j) $y = \frac{1+\sin x}{x+\cos x}$
k) $y = \frac{\sec x}{1+\tan x}$
l) $y = \frac{1-\sec x}{1+\tan x}$
m) $y = \frac{\sin x}{x^2}$
n) $y = \csc x(x + \cot x)$
o) $y = xe^x \csc x$
p) $y = x^2 \sin x \tan x$

(3) Find the derivative:

a) $y = \sin 4x$
b) $y = \tan(\sin x)$
c) $y = \sin e^x$
d) $y = \cos x^3$
e) $y = \cos^3 x$
f) $y = \sqrt{\tan x + 1}$
g) $y = e^{\sin x}$
h) $y = e^{\sin 3x}$
i) $y = e^{\sec 3x}$
j) $y = \sin(\tan 2x)$
k) $y = \tan^2(3x)$
l) $y = \sec^2 x \tan^2 x$
m) $y = x \sin \frac{1}{x}$
n) $y = e^{5x} \cos 3x$
o) $y = \sin(\cos(\tan x))$

(4) Find the derivative:

a) $y = \frac{1}{\sin^{-1} x}$
b) $y = \tan^{-1} \sqrt{x}$
c) $y = x \tan^{-1} \sqrt{x}$
d) $y = \sin^{-1}(2x + 1)$
e) $y = \sqrt{\tan^{-1} x}$
f) $y = x\sqrt{tan^{-1}x}$
g) $y = \tan^{-1} \sqrt{\frac{1-x}{1+x}}$
h) $y = \cos^{-1} e^{2x}$

CHAPTER 7

Implicit Differentiation

An explicit function of x is written as $y = f(x)$. For example,

$$y = 2x + 3$$

is an explicity function. But if we write it as

$$2x + y = 3$$

then we are writing it as an implicit function of x. For a simple function like this we can write it explicitly or implicitly, but it may be harder to write other functions explicitly. For example,

$$y^4 + 3y - 4x^3 = 5x + 1$$

cannot be written by isolating y on one side and x on the other. This is an example of an implicit function that cannot be written explicitly. Another example is

$$x^4 + y^2 = 5.$$

The basic strategy is to take derivative with respect to x on both sides and when y is encountered to differentiate as usual, but include $\frac{dy}{dx}$. It is easier done than said. We already saw examples of how to use this technique in the previous chapter. In this chapter we will consider more such problems.

EXAMPLE 7.1. Find the derivatives.

(1) $x^3 + y^2 = 3x + 5$

(2) $4xy^3 - x^2y + x^3 - 5x + 6 = 0$

(3) $y = x^2 \sin y$

Solutions.

(1) $x^3 + y^2 = 3x + 5$

$$\begin{aligned}
\frac{d}{dx}(x^3 + y^2) &= \frac{d}{dx}(3x+5) \\
\frac{d}{dx}x^3 + \frac{d}{dx}y^2 &= \frac{d}{dx}3x + \frac{d}{dx}5 \\
3x^2 + 2y\frac{dy}{dx} &= 3 + 0 \\
2y\frac{dy}{dx} &= -3x^2 + 3 \\
\frac{dy}{dx} &= \frac{-3x^2+3}{2y}
\end{aligned}$$

(2) $4xy^3 - x^2y + x^3 - 5x + 6 = 0$

$$\begin{aligned}
\frac{d}{dx}(4xy^3 - x^2y + x^3 - 5x + 6) &= \frac{d}{dx}0 \\
\frac{d}{dx}4xy^3 - \frac{d}{dx}x^2y + \frac{d}{dx}x^3 - \frac{d}{dx}5x + \frac{d}{dx}6 &= \frac{d}{dx}0 \\
4x\frac{d}{dx}y^3 + y^3\frac{d}{dx}(4x) - [x^2\frac{dy}{dx} - y\frac{d}{dx}x^2] + 3x^2 + 5 &= 0 \\
4x(3y^3)\frac{dy}{dx} + y^3(4) - [x^2\frac{dy}{dx} - y(2x)] + 3x^2 + 5 &= 0 \\
12xy^3\frac{dy}{dx} + 4y^3 - x^2\frac{dy}{dx} + 2xy + 3x^2 + 5 &= 0 \\
\frac{dy}{dx}[12xy^3 - x^2] &= -4y^3 - 2xy - 3x^2 - 5 \\
\frac{dy}{dx} &= \frac{-4y^3 - 2xy - 3x^2 - 5}{12xy^3 - x^2}
\end{aligned}$$

(3) $y = x^2 \sin y$

$$\begin{aligned}
\frac{dy}{dx} &= x^2\frac{d}{dx}\sin y + \sin y\frac{d}{dx}x^2 \\
\frac{dy}{dx} &= x^2\cos y\frac{dy}{dx} + (\sin y)(2x) \\
\frac{dy}{dx}(1 - x^2\cos y) &= 2x\sin y \\
\frac{dy}{dx} &= \frac{2x\sin y}{1 - x^2\cos y}
\end{aligned}$$

□

We can use implicit differentiation to find the derivatives of $f(x) = \ln x$ and $f(x) = b^x$. Many would consider these proofs easier than the previous ones in Chapter 5. So it is worth reviwing the next example even though we saw it earlier.

Example 7.2. Find the derivatives.

(1) $y = \ln x$

(2) $y = b^x$

Solutions.

(1) $y = \ln x$

Since $y = \ln x$ by definition

$$e^y = x$$

Taking derivatives on both sides and using implicit differentiation

$$\begin{aligned} \frac{dy}{dx}e^y &= \frac{d}{dx}x \\ e^y\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{e^y} \\ \frac{dy}{dx} &= \frac{1}{x} \end{aligned}$$

(2) Since $y = b^x$ we can take log on both sides to get

$$\ln y = \ln b^x$$

Using log properties

$$\ln y = x \ln b$$

Taking derivatives on both sides and using implicit differentiation

$$\begin{aligned} \frac{d}{dx}\ln y &= \frac{d}{dx}x \ln b \\ \frac{1}{y}\frac{dy}{dx} &= \ln b \\ \frac{dy}{dx} &= y \ln b \\ \frac{dy}{dx} &= b^x \ln b \end{aligned}$$

□

The technique of taking log on both sides requires some explanation. Suppose f is a one-to-one function, then we can compose f with another one-to-one function g to get $g \circ f$. This is what we are doing when we take log on both sides. The log function is one-to-one. [1] This means

$$m = n \iff log_b m = log_b n$$

The next set of problems involve a new function

$$f(x) = x^x$$

whose graph is shown in Figure 7.0.1. This is not a power function nor an exponential function. If $x = 1$, then $y = 1^1 = 1$. Between 0 and 1 the function dips and past 1 it rises. The function is not defined at 0. When $x < 0$ the situation is more complicated. For example, if $x = -2$, $y = (-2)^(- 2) = \frac{1}{(-2)^2} = \frac{1}{4}$. If $x = -3$, then $y = (-3)^(- 3) = \frac{1}{(-3)^3} = \frac{1}{-27}$. However, if $x = \frac{1}{2}$, then $y = (-0.5)^(- 0.5) = \frac{1}{(-0.5)^{-0.5}}$, which is a complex number. It is hard to write the domain since the real and complex values of the function are intermingled. Figure 7.0.1 shows both the real and complex portions of the graph.

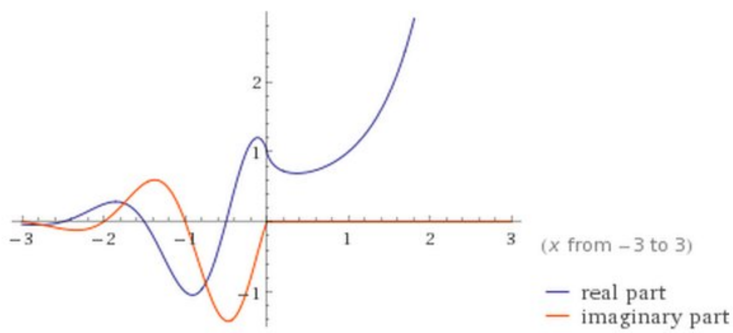


FIGURE 7.0.1. $y = x^x$

EXAMPLE 7.3. Find the derivatives.

(1) $y = x^x$

(2) $y = x^{\sqrt{x}}$

(3) $y = x^{\cos x}$

[1] A function doesn't have to be one-to-one, but then things get a bit messy and must be analyzed on a case-by-case basis. For example, when solving radical equations, we square both sides and the squaring function is not one-to-one. As a result we get extraneaous solutions.

The strategy here is to take log on both sides and to use the log property

$$\log_b x^t = t \log_b x.$$

Solution.

(1) $y = x^x$

Starting with $y = x^x$, take ln on both sides to get

$$\begin{aligned}
\ln y &= \ln x^x \\
\ln y &= x \ln x \\
\frac{1}{y}\frac{dy}{dx} &= x\frac{d}{dx}\ln x + (\ln x)\frac{d}{dx}x \\
\frac{1}{y}\frac{dy}{dx} &= x\frac{1}{x} + (\ln x)(1) \\
\frac{dy}{dx} &= y[1 + \ln x] \\
\frac{dy}{dx} &= x^x[1 + \ln x]
\end{aligned}$$

(2) $y = x^{\sqrt{x}}$

$$\begin{aligned}
\ln y &= \ln x^{\sqrt{x}} \\
\ln y &= \sqrt{x}\ln x \\
\frac{1}{y}\frac{dy}{dx} &= \sqrt{x}\frac{d}{dx}\ln x + (\ln x)\frac{d}{dx}\sqrt{x} \\
\frac{1}{y}\frac{dy}{dx} &= \sqrt{x}\frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}} \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{\sqrt{x}} + (\ln x)\frac{1}{2\sqrt{x}} \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{\sqrt{x}}[1 + \frac{\ln x}{2}] \\
\frac{dy}{dx} &= \frac{y}{\sqrt{x}}[1 + \frac{\ln x}{2}] \\
\frac{dy}{dx} &= \frac{x^{\sqrt{x}}}{\sqrt{x}}[1 + \frac{\ln x}{2}]
\end{aligned}$$

(3) $y = x^{\cos x}$

$$
\begin{aligned}
lny &= x^{\cos x} \\
\ln y &= \ln x^{\cos x} \\
\ln y &= \cos x \ln x \\
\frac{1}{y}\frac{dy}{dx} &= \cos x \frac{d}{dx} \ln x + (\ln x)\frac{d}{dx} \cos x \\
\frac{1}{y}\frac{dy}{dx} &= (\cos x)(\frac{1}{x}) + (\ln x)(-\sin x) \\
\frac{dy}{dx} &= y[\frac{\cos x}{x} + (\ln x)(-\sin x)] \\
\frac{dy}{dx} &= x^{\cos x}[\frac{\cos x}{x} + (\ln x)(-\sin x)]
\end{aligned}
$$

□

EXAMPLE 7.4. Find the second derivatives.

(1) $x^2 + y^2 = 10$

(2) $x^4 + y^4 = 10$

Previously we did not simplify the derivative once we found them since the emphasis was on knowing and using the formulas and rules correctly. Now it makes sense to simplify the first derivative as much as possible before finding the second derivative. So the solutions of these problems can get quite long.

Solution.

(1)

$$x^2 + y^2 = 10$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}10$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= -\frac{y\frac{d}{dx}x - x\frac{d}{dx}y}{y^2} \\
&= -\frac{y - x\frac{dy}{dx}}{y^2} \qquad \text{Put } \frac{dy}{dx} = -\frac{x}{y} \\
&= -\frac{y - x\frac{-x}{y}}{y^2} \\
&= -\frac{y^2 + x^2}{y^3} \qquad \text{Put } x^2 + y^2 = 10 \\
&= -\frac{10}{y^3}
\end{aligned}$$

(2)

$$x^4 + y^4 = 10$$

$$\frac{d}{dx}x^4 + \frac{d}{dx}y^4 = \frac{d}{dx}10$$

$$4x^3 + 4y^3\frac{dy}{dx} = 0$$

$$x^3 + y^3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^3}{y^3}$$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= -\frac{y^3\frac{d}{dx}x^3 - x^3\frac{d}{dx}y^3}{y^6} \\
&= -\frac{y^3 3x^2 - x^3 3y^2\frac{dy}{dx}}{y^6} \qquad \text{Put } \frac{dy}{dx} = -\frac{x^3}{y^3} \\
&= -\frac{3x^2y^3 - 3x^3y^2\frac{-x^3}{y^3}}{y^6} \\
&= -\frac{3x^2y^3 + \frac{3x^6}{y}}{y^6} \\
&= -\frac{3x^2y^4 + 3x^6}{y^7} \\
&= -\frac{3x^2(y^4 + x^4)}{y^7} \qquad \text{Put } x^4 + y^4 = 10 \\
&= -\frac{3x^2(10)}{y^7} \\
&= -\frac{30x^2}{y^3}
\end{aligned}$$

□

This chapter concludes the topic of derivatives and how to find them. In the next chapter we will learn a method for finding limits based on the derivative. In subsequent chapters we will see some fascinating real world problems.

Practice Problems

(1) Find the derivatives.

a) $x^2 + y^2 = 36$

b) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 9$

c) $x^3 - xy + y^2 = 4$

d) $x^3y^3 - y = x$

e) $x^3 - 3x^2y + 2xy^2 = 12$

f) $\sin x + 2\cos 2y = 1$

g) $\sin x = x(1 + \tan y)$

h) $y = \sin(xy)$

(2) Find the second derivative.

a) $y = 4x^{\frac{3}{2}}$

b) $y = \frac{x}{x-1}$

c) $y = 3\sin x$

d) $y = \sec x$

e) $9x^2 + y^2 - 9$

f) $x^3 + y^3 = 1$

g) $x^4 + y^4 = 16$

h) $\sqrt{x} + \sqrt{y} = 1$

CHAPTER 8

L'Hôpital's Rule

L'Hôpital's Rule named after French mathematician Guillaume de L'Hôpital (1661- 1704) is an incredibly simple and useful way of finding the limits of complicated functions. It depends on the derivative, which is why we waited so long to see it. L'Hôpital's published the rule in his 1696 book *Analysis of the Infinitely Small for the Understanding of Curved Lines*, which was the first textbook on differential calculus. Actually, Swiss mathematician Johann Bernoulli (1667 âĂŞ 1748), one of Euler's teachers, described this method in response to a question posed by L'Hôpital, and in his book L'Hôpital credits Bernoulli for this theorem, but subsequently everyone calls it L'Hôpital's rule.

FIGURE 8.0.1. Guillaume de L'Hôpital

Theorem 8.1. (L'Hôpital's Rule) Let f and g be functions that are differentiable on an open interval containing a, except possibly at a itself. Let $c < a < d$ and suppose that $g(x) \neq 0$ for every x in (c, d). If the limit of $\frac{f(x)}{g(x)}$ as x approaches a produces the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, providing it exists or is infinite.

We will see the proof of this result in Chapter 12. But we can begin using it right away. Note that L'Hôpital Rule applies to functions of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. We will see later that it can also be used for other indeterminate forms such as $0 \times \infty$ and 1^{∞}.

EXAMPLE 8.2. Use L'Hôpital's Rule to find the limits, if possible.

(1) $\lim_{x\to 1} \frac{\ln x}{x-1}$

(2) $\lim_{x\to\infty} \frac{e^x}{x^2}$

(3) $\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$

(4) $\lim_{x\to 0} \frac{\tan x - x}{x^3}$

Solution.

(1) Observe that $\lim_{x\to 1} \frac{\ln x}{x-1}$ has $\frac{0}{0}$ form. So

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1$$

(2) Observe that $\lim_{x\to\infty} \frac{e^x}{x^2}$ has $\frac{\infty}{\infty}$ form. So

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x}$$

Now we can apply L'Hôpital Rule's again since $\lim_{x\to\infty} \frac{e^x}{2x}$ again has $\frac{\infty}{\infty}$ form. So

$$\lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

.

Putting it all together (without using so many words) we get

$$\begin{aligned} \lim_{x\to\infty} \frac{e^x}{x^2} &= \lim_{x\to\infty} \frac{e^x}{2x} \\ &= \lim_{x\to\infty} \frac{e^x}{2x} \\ &= \lim_{x\to\infty} \frac{e^x}{2} \\ &= \infty \end{aligned}$$

(3) Observe that $\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$ has $\frac{\infty}{\infty}$ form. So

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} &= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-\frac{2}{3}}} \\ &= \lim_{x\to\infty} \frac{3}{xx^{-\frac{2}{3}}} \\ &= \lim_{x\to\infty} \frac{3}{\sqrt[3]{x}} \\ &= 0\end{aligned}$$

(4) Observe that $\lim_{x\to 0} \frac{\tan x - x}{x^3}$ has $\frac{0}{0}$ form.

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x - x}{x^3} &= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \\ &= \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} \\ &= \left(\frac{1}{3}\lim_{x\to 0} \sec^2 x\right) \cdot \left(\lim_{x\to 0} \frac{\tan x}{x}\right) \\ &= \frac{1}{3} \cdot \lim_{x\to 0} \frac{\sec^2 x}{1} \\ &= \frac{1}{3} \cdot 1 \\ &= \frac{1}{3}\end{aligned}$$

□

Next, let us consider problems of the type $f(x)g(x)$ where one function tends to zero and the other tends to $\pm\infty$. In such cases, it isn't clear whether $f(x)$ or $g(x)$ will dominate in the limit. This kind of indeterminate form is of type $0 \cdot \infty$. Our strategy is to write $f \cdot g$ as $\frac{f}{\frac{1}{g}}$ or $\frac{g}{\frac{1}{f}}$ so that it will have either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

EXAMPLE 8.3. Find the limits.

(1) $\lim_{x\to 0^+} x \ln x$

Solution.

(1) We write $\lim_{x\to 0^+} x\ln x$ as $\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$ which is in the $-\frac{\infty}{\infty}$ form. So L'Hôpital's Rule applies and we get

$$\begin{aligned}\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \\ &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ &= \lim_{x\to 0^+} -x \\ &= 0\end{aligned}$$

□

Lastly, consider function of the indeterminate form 1^∞. In this case, we take *log* on both sides and turn it into a function of the form $f(x)g(x)$ and use the previous approach.

Example 8.4. Find the limits.

(1) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$

(2) $\lim_{x\to 0^+} (\sin x)^x$

Solutions.

(1) Let $y = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$ and take ln on both sides

$$\begin{aligned}\ln y &= \ln \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x \\ &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \\ &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\ &= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \text{ (by L'Hôpital's Rule)} \\ &= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} \\ &= 1\end{aligned}$$

Since $\ln y = 1$, it follows that $y = e$.

(2) Let $y = \lim_{x\to 0^+} (\sin x)^x$ and take ln on both sides.

$$\begin{aligned}
\ln y &= \lim_{x\to 0^+} x\ln(\sin x)\\
&= \lim_{x\to 0^+} \frac{\ln\sin x}{\frac{1}{x}} \text{ (which has } \frac{\infty}{\infty} \text{ form)}\\
&= \lim_{x\to 0^+} \frac{\frac{1}{\sin x}\cos x}{-\frac{1}{x^2}}\\
&= \lim_{x\to 0^+} \frac{\cot x}{-\frac{1}{x^2}}\\
&= \lim_{x\to 0^+} \frac{-x^2}{\tan x} \text{ (which has } \frac{0}{0} \text{ form)}\\
&= -\lim_{x\to 0^+} \frac{2x}{\sec^2 x}\\
&= 0
\end{aligned}$$

Since $\ln y = 0$, it follows that $y = e^0 = 1$.

□

Practice Problems

Find the limit if it exists.

(1) $\lim_{x\to 1} \frac{x^2-1}{x^2-x}$

(2) $\lim_{x\to 1} \frac{x^9-1}{x^5-1}$

(3) $\lim_{x\to 2} \frac{x^2+x+6}{x-2}$

(4) $\lim_{x\to \frac{\pi}{2}^+} \frac{\cos x}{1-\sin x}$

(5) $\lim_{x\to 0} \frac{\sin 4x}{\tan 5x}$

(6) $\lim_{x\to 0} \frac{e^x-1}{x^3}$

(7) $\lim_{x\to 0} \frac{e^{3x}-1}{x}$

(8) $\lim_{x\to \infty} \frac{x+x^2}{1-2x^2}$

(9) $\lim_{x\to \infty} \frac{\ln x}{x}$

(10) $\lim_{x\to 0^+} \frac{\ln x}{x}$

(11) $\lim_{x\to\infty} \frac{e^x}{x^3}$

(12) $\lim_{x\to 0}(1-x)^{\frac{1}{x}}$

(13) $\lim_{x\to 0+}(\tan x)^{x^2}$

(14) $\lim_{x\to\infty}(\ln x)^{\frac{1}{x}}$

(15) $\lim_{x\to 0+}(1+\sin 4x)^{\cot x}$

(16) $\lim_{x\to\infty}(e^x + x)^{\frac{1}{x}}$

CHAPTER 9

Related Rates

Related rates is the phrase used to describe the situation when two or more related variables are changing with respect to time. The rate of change, as mentioned earlier, is another expression for the derivative. For example, suppose water is draining from a cone with a hole in the bottom. We have the fixed height of the cone and the radius of its base and the changing height of the water and radius of its base. At the begining they are the same. After, for example, 10 seconds of water draining, the height of the water in the cone changes and will continue changing until all the water is drained out of the cone. The volume of a cone is given by the formula

$$V = \frac{\pi}{3}r^2h$$

shown in Figure 9.0.1 To find the volume of a cone and other geometric objects, type "Volume of a cone" in Google Search and the interactive Google Calculator shown in Figure 9.0.1 will come up.

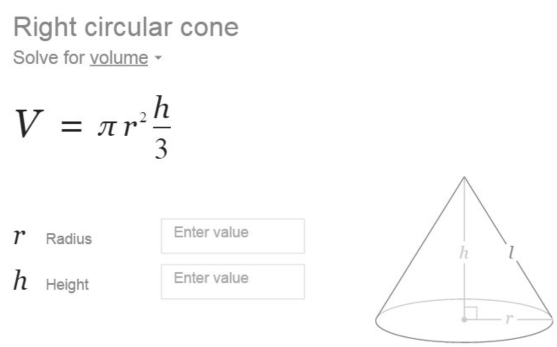


FIGURE 9.0.1. $V = \frac{\pi}{3}r^2h$

There are three variables in this formula, V, r, and h, all changing at different, but related, rates with respect to t. Suppose we differentiate both sides of the formula implicitly with respect to t. Then by the product rule we get

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2\frac{dh}{dt} + h\frac{d}{dt}r^2\right]$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right]$$

The above equation is an equation with derivatives. It describes how the rate of change of volume V is related to the rates of change of height h and radius r. If we are given r, $\frac{dr}{dt}$, h

and $\frac{dh}{dt}$, then we can find $\frac{dV}{dt}$. What makes some of these problems slightly tricky is that we are usually given one or two terms and we have to derive the rest. In any case, the cone is not the best first example to begin with. We will start with some easier examples. But first we must review some formulas from geometry.

(1) Volume of a cylinder $= \pi r^2 h$, where r is the radius of the base and h is the height;

(2) Surface area of a cylinder $= 2\pi r(r + h)$;

(3) Volume of a sphere $= \frac{4}{3}\pi r^3$;

(4) Surface area of a sphere $= 4\pi r^2$, where r is the radius of the base and h is the height;

(5) Volume of a cone $= \frac{\pi}{3}\pi r^2 h$, where r is the radius of the base and h is the height;

(6) Surface area of a cone $= \pi r(r + \sqrt{r^2 + h^2})$.

The area of a circle and the volume and surface area of a cylinder were known well before Archimedes (c. 287 BC - 212 BC). In his paper *Sphere and Cylinder* Archimedes gave the formulas for the volume and surface area of a sphere. He showed that the ratio of the volume of a sphere to the volume of the cylinder that contains it is 2:3 and the ratios of the surface areas is also 2:3. He was so pleased with this formula that he wanted a circle inscribed in a cylinder and the ratio 2:3 inscribed on his tomb. Archimedes was killed by a thoughtless Roman soldier who had no idea who he was, as he was thoughtfully drawing circles on the sand (see Figure 9.0.2). [1] His well-known last words were "Don't disturb my circles." With time the location of his tomb was forgotton. Roman Philosopher Marcus Tullius Cicero (106 BC - 43 BC) discovered his tomb over a hundred years later and recognized it by the inscription of a circle inside a cylinder.

Example 9.1. (Pebble in Pond) A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate of 2 foot per second. When the radius is 6 feet, at what rate is the area of the disturbed water changing?

Solution. The formula for the area of the disturbed water is

$$A = \pi r^2$$

Differentiating implicitly with respect to time t

$$\frac{d}{dt}A = \frac{dr}{dt}\left[\pi r^2\right]$$
$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

[1] An illustration from the page of an unidentified book in the files of the Print and Picture Collection of the Free Library of Philadelphia. `http://www.math.nyu.edu/ crorres/Archimedes/Death/DeathIllus.html`

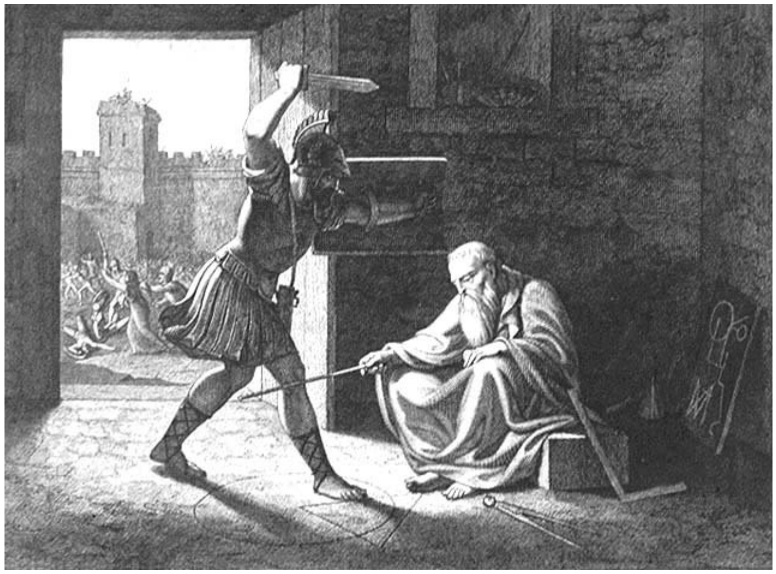


FIGURE 9.0.2. Tod des Archimedes

Since $r = 6$ and $\frac{dr}{dt} = 2$ feet per second,

$$\frac{dA}{dt} = 2\pi(6)(2) = 24\pi$$

Therefore the rate of change of the area is 24π square feet per second. $\square$

EXAMPLE 9.2. (Inflating Balloon) Air is being pumped into a spherical balloon at a rate of 5 cubic feet per minute. What is the rate of change of the radius when the radius is 2 feet?

Solution. The formula for the volume of the inflating baloon is

$$V = \frac{4}{3}\pi r^3$$

Differentiating implicitly with respect to time t

$$\frac{d}{dt}V = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$
$$\frac{dV}{dt} = \frac{4}{3}3\pi r^2\frac{dr}{dt}$$

Since $r = 2$ and $\frac{dV}{dt} = 5$

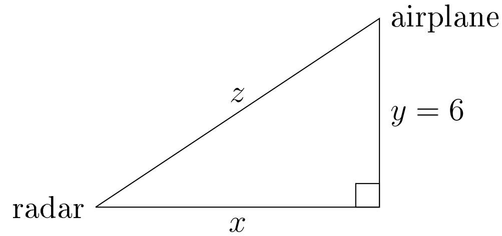
$$5 = 4\pi(2^2)\frac{dr}{dt}$$

$$5 = 16\pi \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{5}{16\pi}$$

Therefore the rate of change of the radius is $\frac{5}{16\pi}$ square feet per second. □

EXAMPLE 9.3. (Flying Airplane) An airplane is flying over a radar tracking station at a height of 6 miles. Suppose the distance is decreasing at a rate of 400 miles per hour. What is the velocity of the plane when the distance is 10 miles?



Solution. Since the triangle is a right triangle, by the Pythagorean Theorem

$$x^2 + y^2 = z^2$$

where x is the horizontal distance, y is the vertical distance and the hypotenuse z is the distance of the airplane from the station. In this problem, x and z are changing, but the height of the plane is fixed at $y = 6$. So the equation is

$$x^2 + 6^2 = z^2$$

Differentiating implicitly with respect to time t

$$\frac{d}{dt}x^2 + \frac{d}{dt}6^2 = \frac{d}{dt}z^2$$

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

$$x\frac{dx}{dt} = z\frac{dz}{dt}$$

We are given that $y = 6$ (which we already used) and $z = 10$. By the Pythagorean Theorem we can find x as

$$x^2 = z^2 - y^2 = 10^2 - 6^2 = 64$$

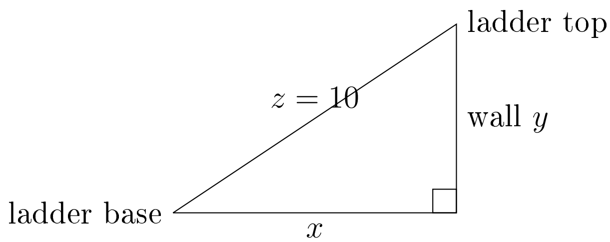
So $x = 8$. We are also given $\frac{dz}{dt} = -400$ (the negative sign stands for decreasing distance).

$$8\frac{dx}{dt} = 10(-400)$$

$$\frac{dx}{dt} = -500$$

The airplane is **approaching** the radar station at the rate of 500 miles per hour. □

EXAMPLE 9.4. (Sliding Ladder) A ladder 10 feet long rests against a vertical wall. If the base of the ladder slides away from the wall at a rate of 1 foot per second, how fast is the top of the ladder sliding down the wall when the bottom is 6 feet away from the wall?



Solution. Since the triangle is a right triangle, by the Pythagorean Theorem

$$x^2 + y^2 = z^2$$

where x is the horizontal distance of the ladder from the wall, y is the wall, and the hypotenuse z is the length of the ladder. In this problem, x and y are changing, but the length of the ladder is fixed at $z = 10$. So the equation is

$$x^2 + y^2 = 10^2$$

Differentiating implicitly with respect to time t

$$\begin{aligned} \frac{d}{dt}x^2 + \frac{d}{dt}y^2 &= \frac{d}{dt}100 \\ 2x\frac{dx}{dt} + 2y\frac{dz}{dt} &= 0 \\ x\frac{dx}{dt} + y\frac{dz}{dt} &= 0 \end{aligned}$$

We are given that $z = 10$ (which we already used) and $x = 8$. By the Pythagorean Theorem we can find y as

$$y^2 = z^2 - x^2 = 10^2 - 8^2 = 36$$

So $y = 6$. We are also given $\frac{dx}{dt} = 1$. We must find $\frac{dx}{dt}$.

$$\begin{aligned} 8(1) + 6\frac{dy}{dt} &= 0 \\ \frac{dy}{dt} &= -\frac{8}{6} \end{aligned}$$

The ladder is sliding **down** the wall at the rate of $1.\bar{3}$ feet per second. □

Example 9.5. (Moving Car) Car A is traveling west at 50 miles per hour and car B is traveling north at 60 miles per hour. Both are headed for the intersection of two roads. At what rate are the cars approaching each other when car A is 0.3 miles from the intersection and car B is 0.4 miles from the intersection?

Solution. Since the triangle is a right triangle, by the Pythagorean Theorem

$$x^2 + y^2 = z^2$$

where x is the horizontal distance west travelled by Car A, y is the vertical distance north travelled by the Car B, and the hypotenuse z is the distance between the cars, which is shrinking.

$$x^2 + y^2 = z^2$$

Differentiating implicitly with respect to time t

$$\frac{d}{dt}x^2 + \frac{d}{dt}y^2 = \frac{d}{dt}z^2$$

$$2x\frac{dx}{dt} + 2y\frac{dz}{dt} = 2z\frac{dz}{dt}$$

$$x\frac{dx}{dt} + y\frac{dz}{dt} = z\frac{dz}{dt}$$

We are given that $x = 0.3$ and $y = 0.4$. By the Pythagorean Theorem we can find z as

$$z^2 = x^2 + y^2 = 0.3^2 + 0.4^2 = 0.25$$

So $z = 0.5$. We are also given $\frac{dx}{dt} = -50$ miles per hour and $\frac{dy}{dt} = -60$ miles per hour. The signs are negative because the cars are "approaching" the intersection. In other words the distance between the cars and the intersection is decreasing.

$$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$$

$$0.3(-50) + 0.4(-60) = 0.5\frac{dz}{dt}$$

$$\frac{dz}{dt} = -78$$

The cars are approaching each other at 78 miles per hour. □

Example 9.6. (Water Tank) A water tank has the shape of an inverted cone with a base of radius of 2 meters and height 4 meters. If water is being pumped into the tank at a rate of 2 cubic meters per minute, find the rate at which the water level is rising when the water is 3 meters deep.

Solution. The volume of a cone is

$$V = \frac{\pi}{3} r^2 h$$

where r is the radius of the base, h is the height of the cone. We are given $r = 2$ and $h = 4$. Based on this we can conclude $r = \frac{h}{2}$. (Note this is true only for this particular problem.)

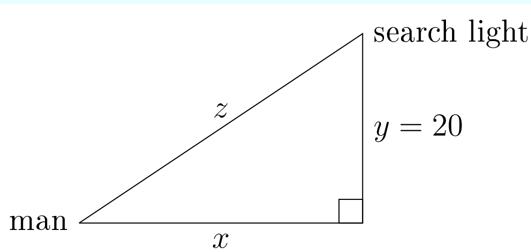
$$\begin{aligned} V &= \frac{\pi}{3} r^2 h \\ &= \frac{\pi}{3} (\frac{h}{2})^2 h \\ &= \frac{\pi h^3}{12} \end{aligned}$$

Differentiate implicitly with respect to t

$$\begin{aligned} \frac{dV}{dt} &= \frac{3\pi}{12} h^2 \frac{dh}{dt} \\ 2 &= \frac{\pi}{4} h^2 \frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{8}{9\pi} \end{aligned}$$

The rate at which the water level is rising is $\frac{8}{9\pi}$ meters per minute. □

Example 9.7. A man walks along a straight path at a speed of 4 feet per second toward a pole with a searchlight on top. The height of the searchlight is 20 feet and it is kept focused on the man rotating downward as the man moves. At what rate is the searchlight rotating when the man is 15 feet from the pole?



Solution. From the above triangle

$$x^2 + y^2 = z^2$$

where x is the distance of the man from the pole, $y = 20$ is the height of the searchlight (which is fixed) and the hypotenuse z is the distance of the man from the search light. The equation to be used here is

$$\tan\theta = \frac{x}{20}$$

since we want the angle of rotation of the searchlight and it rotates downwards toward the man as he approaches the pole. Differentiating implicitly with respect to t gives

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt}$$

In order to find $\frac{d\theta}{dt}$ we need $\sec\theta$ and $\frac{dx}{dt}$. We are given that $\frac{dx}{dt} = -4$ feet per second. The negative sign appears since the man is approaching the pole. But we have to find $\sec\theta$ from the available information. Using the fact that $x = 15$ and $y = 20$, by the Pythagorean Theorem

$$z = x^2 + y^2 = 15^2 + 20^2 = 625$$

So $z = 25$ and from the right triangle we get

$$\cos\theta = \frac{20}{25} = \frac{4}{5}$$

so

$$\sec\theta = \frac{5}{4}.$$

Putting it together we get

$$\begin{aligned}\frac{dx}{dt} &= 20\sec^2\theta\frac{d\theta}{dt}\\ -4 &= 20\left(\frac{5}{4}\right)^2\frac{d\theta}{dt}\\ \frac{d\theta}{dt} &= -\frac{4}{20}\left(\frac{4}{5}\right)^2\\ &= -\frac{16}{125}\end{aligned}$$

The searchlight is rotating at a rate of $-\frac{16}{125}$ radians per second. □

There are two things to note in the above problem. First the angle is negative since the searchlight is rotating downward. Moreover, the number is angle measure in radians. In degrees the angle is

$$-\frac{16}{125}\times\frac{180}{\pi}\approx -7.34°$$

Second we could have used $x = 15$ and $y = 20$ to write

$$\tan\theta = \frac{x}{y} = \frac{15}{20}$$

and find $\theta = 36.87°$ using the inverse tangent key on a calculator. Then we could put $\theta = 36.87°$ in the related rates equation

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt}$$

As you can see from the last two problems, if all the information is not directly given in the statement of the problem itself, then the problem requires more thought. These sorts of multi-step problems, where you have to figure out one thing and use it to get the next thing, and so on, are typical of real-world problems. However, the above problems are contrived like cardboard cut-outs of real-world problems. It is worth doing a few of these contrived problems to master the techniques and leave it at that.

Practice Problems

(1) A pebble is dropped in a calm pond causing ripples to form in concentric circles. The radius of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the area of disturbed water changing?

(2) Air is pumped into a spherical balloon at a rate of 5 cubic feet per minute. find the rate of change of the radius when the radius is 3 feet.

(3) Air is pumped into a spherical balloon so that the volume increases at a rate of 100 cubic centimeters per second. How fast is the radius of the balloon increasing when the diameter is 50 centimeters?

(4) If a snowball melts so that its surface area decreases at a rate of 2 square centimeters per minute, find the rate at which the diameter decrease when the diameter is 10 centimeters.

(5) All edges of a cube are expanding at a rate of 3 centimeters per second. How fast is the volume changing when each edge is 10 centimeters?

(6) A water tank has the shape of an inverted circular cone with base radius 2 meters and height 4 meters. If water is being pumped into the tank at a rate of 2 cubic meters per minute, find the rate at which the water level is rising when the water is 3 meters deep.

(7) Gravel is being dumped from a conveyor belt at a rate of 30 cubic feet per minute and forms a conical pile whose base diameter and height are equal. How fast is the height of the pile increasing when the pile is 10 feet high?

(8) Sand is falling off a conveyor belt onto a conical pile at a rate of 10 cubic feet per minute. The diameter of the base of the cone is three times the height. At what rate is the height of the pile changing when the pile is 15 feet high?

(9) A conical tank with vertex down is 10 feet across at the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

(10) An airplane is flying over a radar tracking station at a height of 6 miles. If the distance between the plane and the station is decreasing at a rate of 400 miles per hour, when the distance is 10 miles, what is the speed of the plane?

(11) A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 foot per second, how fast is the top of the ladder sliding down the wall, when the bottom is 6 feet from the wall?

(12) Car A is traveling west at 50 miles per hour and car B is traveling north at 60 miles per hour headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 miles and car B is 0.4 miles from the intersection?

(13) Two cars start moving from the same point. One travels south at 60 miles per hour and the other travels west at 25 miles per hour. At what rate is the distance between the cars increasing 2 hours later?

CHAPTER 10

Maxima and Minima

Arguably the whole purpose of derivatives is to determine the maximum and minimum of functions. Take a look at Figure 10.0.1. It shows the graph of the quadratic function $f(x) = -x^2 + 8x + 20$. Reading the graph from left to right, observe how the graph rises (increases) on the left reaches the highest point at the vertex and then falls (decreases) on the right. Notice how the graph is rapidly increasing at first and then as it approaches the vertex, the rate of increase although still positive decreases. The rate of change is the slope of the tangent line. Imagine small tangent lines draw at various points on the graph. At the vertex the tangent line becomes horizontal (the slope is zero). Then the slope of the tangent line gradually becomes negtive, slowly at first, then faster and faster.

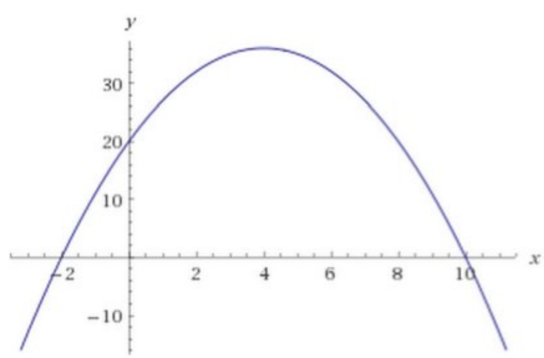


FIGURE 10.0.1. $f(x) = -x^2 + 8x + 20$

We have an easy formula for finding the vertex of a quadratic function $f(x) = ax^2 + bx + c$. The x-coordinate of the vertex is

$$x = -\frac{b}{2a}.$$

So the x-coordinate of vertex of $f(x) = -x^2 + 8x + 20$ is

$$x = -\frac{8}{2(-1)} = 4.$$

The y-coordinate of the vertex is

$$f(4) = -(4)^2 + 8(4) + 20 = -16 + 32 + 20 = 36.$$

Thus, the vertex of $f(x) = -x^2 + 8x + 20$ is $(4, 36)$. But what if the functions are $f(x) = x^4 - 8x^2 + 16$ or $g(x) = x^5 - 8x^3 + 16x$ whose graphs are shown in Figure 10.0.2. We don't

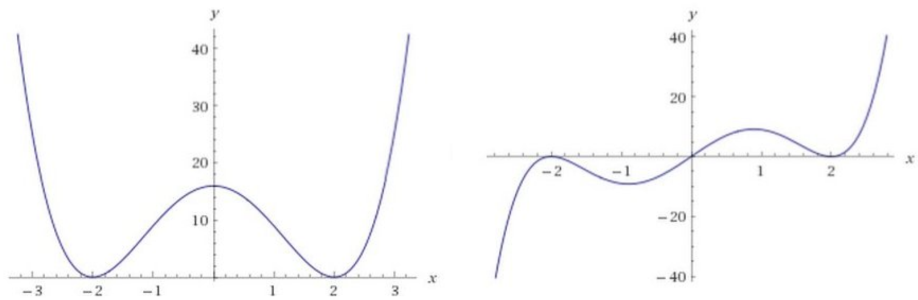


FIGURE 10.0.2. $f(x) = x^4 - 8x^2 + 16$ (left) and $g(x) = x^5 - 8x^3 + 16x$ (right)

have formulas for the highest and lowest points like we did for quadratic functions. You may think that zooming into the graph will help, and it might work sometimes, but not always. In his article "Change we can believe in" Steven Strogatz describes local maxima and minima as follows:

> There's a more general principle at work here - things always change slowest at the top or the bottom. It's especially noticeable here in Ithaca. During the darkest depths of winter, the days are not just unmercifully short; they barely improve from one to the next. Whereas now that spring is popping, the days are lengthening rapidly. All of this makes sense. Change is most sluggish at the extremes precisely because the derivative is zero there. Things stand still, momentarily. This zero-derivative property of peaks and troughs underlies some of the most practical applications of calculus. It allows us to use derivatives to figure out where a function reaches its maximum or minimum, an issue that arises whenever we're looking for the best or cheapest or fastest way to do something.

Let's take a careful look at a very familiar function, namely, $f(x) = x^2$ whose graph is shown in Figure 10.0.3 alongside its derivative $f'(x) = 2x$ and second derivative $f''(x) = 2$. There is a close relationship between the graph of the function and the graph of its first and second derivative. Observe that

- Over the interval $(-\infty, 0)$, f is decreasing and f' is negative;
- Over the interval $(0, \infty)$, f is increasing and f' is negative;
- f is concave up and f'' is positive.

Similarly, the graph of $f(x) = -x^2$ and its derivatives $f'(x) = -2x$ and $f''(x) = 2$ are shown in Figure 10.0.4. Observe that

- Over the interval $(-\infty, 0)$, f is increasing and f' is positive;
- Over the interval $(0, \infty)$, f is decreasing and f' is negative;
- f is concave down and f'' is positive.

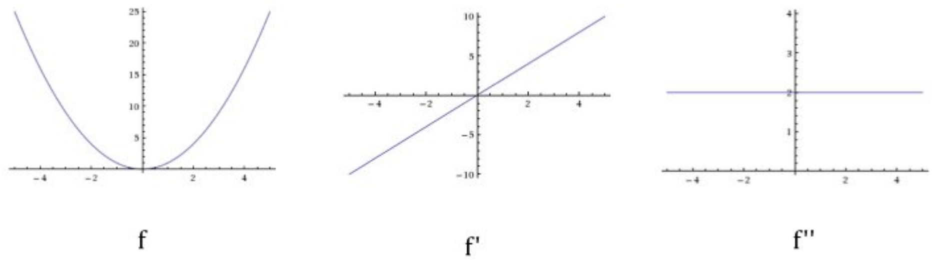


FIGURE 10.0.3. $f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$

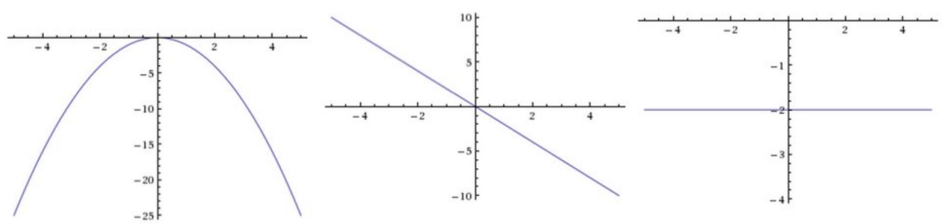


FIGURE 10.0.4. $f(x) = -x^2$, $f'(x) = -2x$, $f''(x) = -2$

When the function is increasing, the slope of the tangent line at each point on the increasing portion is positive. So the derivative (which is the slope of the tangent line) is positive. When the function is decreasing, slope of the tangent line at each point on the decreasing portion is negative. So the derivative is negative. Thus it would be reasonable to conclude that

- f is increasing on (a, b) if and only if f' is positive on (a, b); and
- f is decreasing on (a, b) if and only if f' is negative on (a, b).

Moreover,

- f is concave up on (a, b) if and only if f'' is positive on (a, b); and
- f is concave down on (a, b) if and only if f'' is negative on (a, b);.

These items are useful for determining the shape of the derivative based on the shape of the graph. For example, consider the functions $f(x) = (x-5)^2$, $g(x) = x^3$, and $h(x) = -|x+4|$ shown in Figure 10.0.5. Observe that

- $f(x) = (x-5)^2$ is decreasing on $(-\infty, 5)$ and increasing on $(5, \infty)$. So f' is negative on $(-\infty, 5)$ and positive on $(5, \infty)$
- $g(x) = x^3$ is increasing on $(-\infty, \infty)$. So g' is positive throughout $(-\infty, \infty)$.
- $h(x) = -|x+4|$ is increasing on $(-\infty, -4)$ and decreasing on $(-4, \infty)$. So h' is positive on $(-\infty, -4)$ and negative on $(-4, \infty)$.

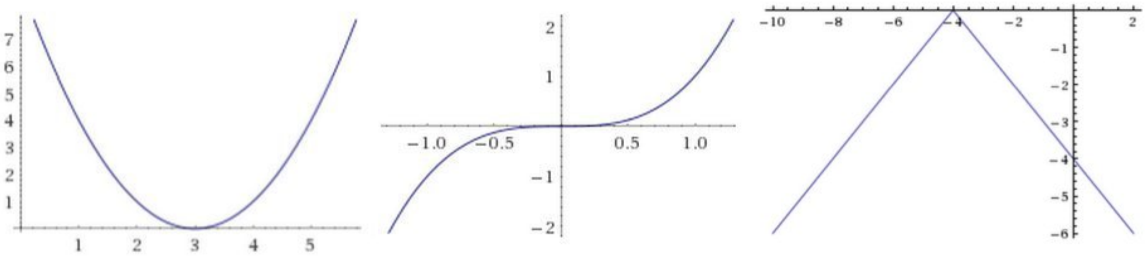


FIGURE 10.0.5. $f(x) = (x-5)^2$, $g(x) = x^3$, $h(x) = -|x+4|$

Note how we can describe quite well the shape of the derivative based soley on the graph of the function. Now let us introduce some terminology to express these ideas.

Definition: Critical Point, Local and Global Maxima and Minima

Let f be a real-valued function and p be a point in the domain of f.

(1) We say p is a **critical point** if either $f'(p) = 0$ or $f'(p)$ is undefined.

(2) We say $f(p)$ is a **local maximum** if $f(x) \leq f(p)$ for all x in an open interval (a, b) containing p.

(3) We say $f(p)$ is a **local minimum** if $f(x) \geq f(p)$ for all x in an open interval (a, b) containing p.

(4) We say $f(p)$ is **the global maximum** if $f(x) \leq f(p)$ for all x in the domain of f.

(5) We say $f(p)$ is **the global minimum** if $f(x) \geq f(p)$ for all x in the domain of f.

Theorem 10.1. (Rolle's Theorem) Suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then $f'(c) = 0$ for at least one $c \in (a, b)$.

Proof: There are three possibilities: $f(x) > f(a)$; $f(x) < f(a)$; and $f(x) = f(a)$.

(1) Suppose $f(x) > f(a)$ for some $x \in (a, b)$. Then the maximum value of f in $[a, b]$ is greater than $f(a)$ or $f(b)$ and occurs at some $c \in (a, b)$. By hypothesis f' exists on (a, b), so $f'(c) = 0$.

(2) Suppose $f(x) < f(a)$ for some $x \in (a, b)$. Then the minimum value of f in $[a, b]$ is less than $f(a)$ or $f(b)$ and occurs at some $c \in (a, b)$. Again, since f' exists on (a, b), $f'(c) = 0$.

(3) Lastly, suppose $f(x) = f(a)$ for all $x \in (a, b)$. Then f is a constant function and $f'(c) = 0$ for all $c \in (a, b)$.

Hence proved. □

Michael Rolle was a 17th century mathematician who published a book titled *Traité d'Algebre* in 1690. It contains the first proof of Rolle's Theorem as well as the first publication of the Gaussian elimination algorithm (both of which were well known to Eastern mathematicians). Rolle's theorem is the basis for the First and Second Derivative Tests for finding local extrema.

Corollary 10.2. (First Derivative Test for Local Extrema) Suppose f is a function with critical point p and f is differentiable on (a, b) except possibly at p.

(1) If f' is positive to the left of p and negative to the right of p, then p is a local maximum of f.

(2) If f' is negative to the left of p and positive to the right of p, then p is a local minimum of f.

Example 10.3. Use the First Derivative Test to find the local extrema, if they exist.

(1) $f(x) = x^3 - 9x^2 - 48x + 52$

(2) $f(x) = \frac{x}{x^2+1}$

(3) $f(x) = x^3$

(4) $f(x) = x^4$

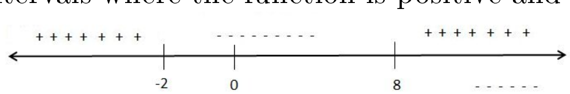
Solution: There are three steps in a local extrema problem: differentiate; find the critical points; use the First Derivative Test.

(1) $f(x) = x^3 - 9x^2 - 48x + 52$ and $f'(x) = 3x^2 - 18x - 48$

To find the critical points solve the equation $f'(x) = 0$:

$$\begin{aligned} 3x^2 - 18x - 48 &= 0 \\ x^2 - 6x - 16 &= 0 \\ (x-8)(x+2) &= 0 \\ x &= 8, -2 \end{aligned}$$

Thus, the critical points are $x = 8$ and $x = -2$. Mark them on a number line and determine the intervals where the function is positive and negative as shown below:



Since f' is positive to the left of -2 and negative to the right of -2, a local maximum occurs at -2. The value of the local maximum is

$$f(-2) = (-2)^3 - 9(-2)^2 - 48(-2) + 52 = 104.$$

Since f' is negative to the left of 8 and positive to the right of 8, a local minimum occurs at 8. The value of the local minimum is

$$f(8) = (8)^3 - 9(8)^2 - 48(8) + 52 = -396.$$

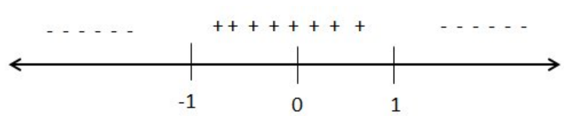
(Note that once the technique is mastered the problem can be solved with fewer words.)

(2)

$$f(x) = \frac{x}{x^2+1}$$
$$f'(x) = \frac{1-x^2}{(x^2+1)^2}$$

Critical Points:

$$1 - x^2 = 0$$
$$x = \pm 1$$



Since f' is positive to the left of -1 and negative to the right of -1, a local maximum occurs at -1. The value of the local maximum is

$$f(-1) = -\frac{1}{2}.$$

Since f' is negative to the left of 1 and positive to the right of 1, a local minimum occurs at 1. The value of the local minimum is

$$f(1) = \frac{1}{2}.$$

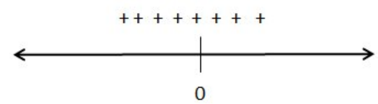
(3)

$$f(x) = x^3$$

$$f'(x) = 3x^2$$

Critical Points:

$$3x^2 = 0$$
$$x = 0$$



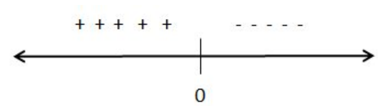
Observe from the above diagram that the sign of f' does not change around 0. So no local extrema occurs at $x = 0$. This can be confirmed by looking at its graph in Figure 10.0.6.

(4)

$$f(x) = x^4$$
$$f'(x) = 4x^3$$

Critical Points:

$$4x^3 = 0$$
$$x = 0$$



Since f' is negative to the left of 0 and positive to the right of 0, a local minimum occurs at 0. The value of the local maximum is $f(0) = 0$.

□

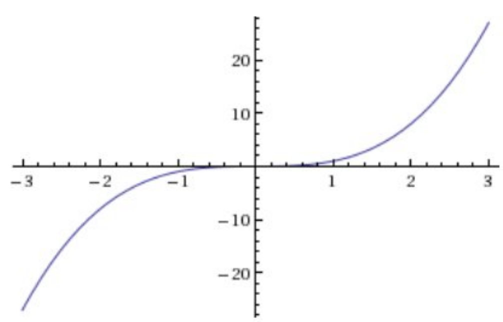


FIGURE 10.0.6. $f(x) = x^3$

The First Derivative Test also makes clear where the function is increasing and where it is decreasing. Graphs with increasing, decreasing, and constant parts have cusps at the end of the constant parts which must be determined. But polynomial functions are straightforward. In Example 10.4 (1) the diagram indicates that $f(x) = x^3 - 9x^2 - 48x + 52$ is decreasing over interval $(-\infty, -2)$ increasing over interval $(-2, 2)$ and decreasing over $(8, \infty)$. The graph of this function is shown in Figure 10.0.7, confirming the analysis of the First Derivative Test.

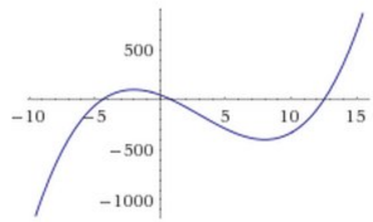


FIGURE 10.0.7. $f(x) = x^3 - 9x^2 - 48x + 52$

Similarly, in 10.4(2) the function $f(x) = \frac{x}{x^2+1}$ is decreasing on $(-\infty, 1) \cup (1, \infty)$ and increasing on $(-1, 1)$. The graph is shown in Figure 10.0.8.

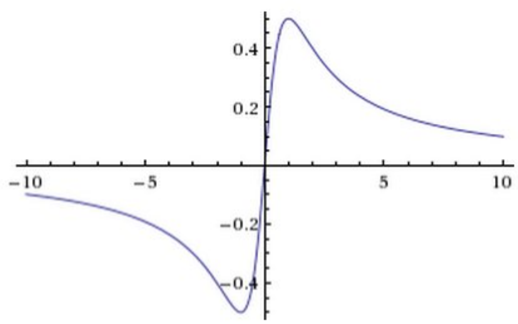


FIGURE 10.0.8. $f(x) = \frac{x}{x^2+1}$

Next, we can use concavity and the second derivative to determine local maxima and minimum

Corollary 10.4. (Second Derivative Test for Local Extrema) Suppose f is a function with critical point p and f is differentiable on (a, b) except possibly at p.

(1) If $f''(p) < 0$ then p is a local maximum of f.

(2) If $f''(p) > 0$, then p is a local minimum of f.

EXAMPLE 10.5. Find the local extrema, if they exist. Use the second derivative test.

(1) $f(x) = x^3 - 9x^2 - 48x + 52$

(2) $f(x) = xe^{-x}$

(3) $f(x) = x + \frac{1}{x}$

Solutions.

(1) $f(x) = x^3 - 9x^2 - 48x + 52$

As we saw in the previous example, $f'(x) = 3x^2 - 18x - 48$ and $f''(x) = 6x - 18$ and f has critical points at $x = -2$ and $x = 8$.

Applying the Second Derivative Test, since $f''(-2) = -30 < 0$, f has a local maximum at -2. Since $f''(8) = 30 > 0$, so f has a local minimum at 8.

(2) $f(x) = xe^{-x}$

$$\begin{aligned} f'(x) &= e^{-x} - xe^{-x} = e^{-x}(1-x) \\ f''(x) &= -e^{-x}(1-x) - e^{-x} = e^{-x}(x-2) \end{aligned}$$

To find the critical points set $e^{-x}(1-x) = 0$ and solve for x. Since the exponential function is never 0, the only way $e^{-x}(1-x)$ can be 0 is when $1 - x = 0$. So $x = 1$. Thus f has a critical point at $x = 1$. Applying the Second Derivative Test, since $f''(1) = -e^{-1} < 0$, f has a local maximum at 1.

(3) $f(x) = x + \frac{1}{x}$

$$\begin{aligned} f'(x) &= 1 - \frac{1}{x^2} \\ f''(x) &= \frac{2}{x^3} \end{aligned}$$

Critical Points:

$$\begin{aligned} 1 - \frac{1}{x^2} &= 0 \\ \frac{1}{x^2} &= 1 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

So f has critical points at $x = \pm 1$. Applying the Second Derivative Test, since $f''(-1) < 0$, f has a local maximum at -1. Since $f''(1) > 0$, f has a local minimum at 1.

□

Arguably the Second Derivative Test is faster to use than the First Derivative Test, but only when the second derivative is easy to find and it doesn't always work. For example, consider the function

$$f(x) = x^4$$

whose local minimal was found to be $x = 0$ using the First Derivative Test. Observe that $f'(x) = 4x^3$ and f has a critical point at $x = 0$. However, $f''(x) = 12x^2$ so $f''(0) = 0$. In this case, the Second Derivative Test failed. The First Derivative Test always works. Moreover, the First Derivative Test also clearly indicates the intervals over which the function is increasing and decreasing as explained earlier.

Corollary 10.6. (Test for concavity) Suppose f is a function differentiable on an open interval (a, b) containing c and suppose $f''(c)$ exists.

(1) If $f''(c) > 0$, then $f'(x)$ is increasing at c, and f is **concave up** at c.

(2) If $f''(c)$ is negative, then $f'(x)$ is decreasing at c, and f is **concave down** at c.

Definition: An **inflection point** is a point on a curve at which a change in the direction of curvature occurs. (The concavity changes.)

Observe that the tangent line to the curve at a point is above the curve for the concave down portion (or convex portion) and below the curve for the concave up portion. The inflection point is where it moves from one side to the other. [1] It can be difficult to determine concavity just by looking at the graph, especially if the concavity is slight. The Test for Concavity gives a precise way of finding the inflection points.

Theorem 10.7. (Test for inflection points) Suppose f is a function differentiable on an open interval (a, b) containing k and suppose $f''(k) = 0$. Then k is an inflection point if and only if f'' changes signs around k.

Example 10.8. Find the inflection points, if they exist.

(1) $f(x) = x^3$

(2) $f(x) = x^3 - 9x^2 - 48x + 52$

(3) $f(x) = x^4 - x$

Solutions. We will find the first and second derivatives and use the Test for Infection points.

[1] The Wikipedia page for "Inflection Points" has an animated gif showing how the tangent line to the curve changes from one side of the curve to the other at the inflection point.

(1) Observe that

$$f(x) = x^3$$
$$f'(x) = 3x^2$$
$$f''(x) = 6x$$

Candidates for inflection points are obtained by setting $f''(x) = 0$ and solving.

$$6x = 0$$
$$x = 0$$

From the diagram above, observe that f'' is negative to the left of 0 and positive to the right of 0. Since the sign of f'' changes using the Test for Inflection Points, we cans ay an inflection point occurs at 0. The infection point is $(0, 0)$.

(2)

$$f(x) = x^3 - 9x^2 - 48x + 52$$
$$f'(x) = 3x^2 - 18x - 48$$
$$f''(x) = 6x - 18$$

Candidates for infection points:

$$6x - 18 = 0$$
$$x = 3$$

From the diagram above, since the sign of f'' changes about $x = 3$, an inflection point occurs at 3. The infection point is $(3, f(3))$.

(3)

$$f(x) = x^4 - x$$
$$f'(x) = 4x^3 - 1$$
$$f''(x) = 12x^2$$

Candidates for infection points:

$$f''(x) = 0$$
$$12x^2 = 0$$
$$x = 0$$

However, f'' does not change sign around $x = 0$. So $x = 0$ is not an inflection point. This function has no inflection point. See the graph of $f(x) = x^4 - x$ in Figure 10.0.9.

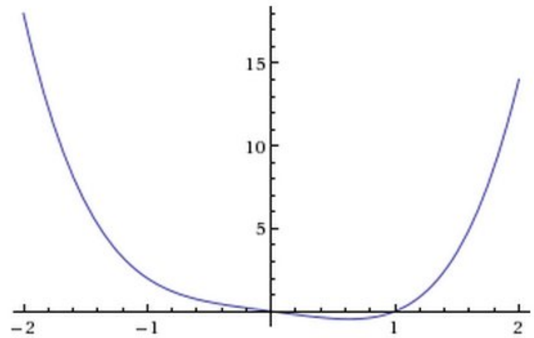


FIGURE 10.0.9. $f(x) = x^4 - x$

□

Finding the intervals over which the function is increasing and decreasing, the local extrema, and the inflection points helps in drawing a good graph. For example take a look at the graph of $f(x) = x^3$ in Figure 10.0.6 and observe the way the curvature changes at $x = 0$. In fact, have you ever wondered why $f(x) = x^3$ looks the way it does? After all we plot by hand just a few points and even though the computer can plot a point for every pixel, it is still just plotting points. There are infinite real numbers between every pair of real numbers, so how do we know something strange does not occur at points to fine to plot. The shape of the graph is governed by the local extrema and inflection points.

Practice Problems

(1) For each function find the following, if they exist:

- i) The critical points
- ii) The local extrema using the first derivative test
- iii) The intervals over which the function is increasing and decreasing
- iv) The local extrema using the second derivative test (if possible)
- v) The inflection points

- a) $f(x) = 5x^2 + 4x$
- b) $f(x) = 3x^2 - 12x + 5$
- c) $f(x) = (x^2 - 1)^3$
- d) $f(x) = x^3 + 3x^2 - 24x$
- e) $f(x) = x^3 + x^2 - x$
- f) $f(x) = 3x^4 + 3x^3 - 6x^2$

g) $f(x) = |3x - 4|$

h) $f(x) = \frac{x}{x^2+1}$

i) $f(x) = \frac{x-1}{x^2-x+1}$

j) $f(x) = \frac{x-1}{x^2+4}$

k) $f(x) = \sqrt{1 - x^2}$

l) $f(x) = x^2 e^{-3x}$

m) $f(x) = \frac{\ln x}{x^2}$

CHAPTER 11

Optimization

In this chapter we take our new found ability to find the maximum and minimum of functions and apply it to all sorts of real world situations. As Euler said

> Since the fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear.

We now know the universe is not as predictable as the golden age mathematicians believed it to be. Nonetheless, many real world phenomena can be reduced to a local extrema problem.

Example 11.1. (Maximizing Area) A farmer has 2400 feet of fence and he wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

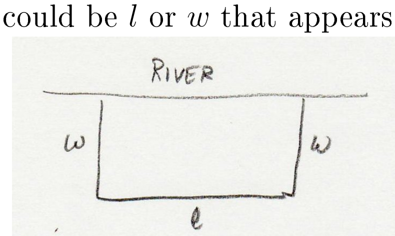
Solution. Suppose l is the length and w is the width of the field. The area A of the field is

$$A = lw$$

and the perimeter P is

$$P = 2w + l$$

since one side is not fenced (it could be l or w that appears once).



The problem specifies that the perimeter is fixed at 2400, which is the amount of fence the farmer has. Using the perimeter formula we get

$$\begin{aligned} P &= 2w + l \\ 2400 &= 2w + l \\ l &= 2400 - 2w. \end{aligned}$$

Substitute the value of l in the area formula to get

$$A = lw = (2400 - 2w)w = 2400w - 2w^2.$$

Since the problem asks for the largest area, the function to maximize is $A = 2400w - 2w^2$. We can use the First Derivative Test or Second Derivative Test. Let's try the Second Derivative Test since it is shorter. The derivative is

$$A' = 2400 - 4w.$$

The critical point is

$$\begin{aligned} A' &= 0 \\ 2400 - 4w &= 0 \\ -4w &= 2400 \\ w &= 600. \end{aligned}$$

The second derivative is

$$A'' = -4.$$

Since the second derivative is negative, A has a maximum at

$$w = 600.$$

The corresponding value of l is

$$l = 2400 - 2w = 2400 - 2(600) = 1200.$$

Therefore, the field with the largest area has $w = 600$ feet and $l = 1200$ feet. □

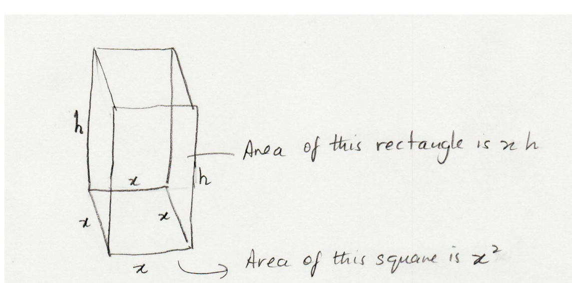
Example 11.2. (Maximizing Volume) An open box having a square base and a surface area of 108 square inches must be constructed. What dimensions will produce a box with maximum volume?

Solution: Suppose the box has a square base of length x and height h. The volume of the box is

$$V = x^2h$$

and the surface area is

$$S = x^2 + 4xh.$$



The problem specifies that the surface area is fixed at 108 square inches. Using the surface area formula

$$\begin{aligned} S &= x^2 + 4xh \\ 108 &= x^2 + 4xh \\ 108 - x^2 &= 4xh \\ h &= \frac{108 - x^2}{4x}. \end{aligned}$$

Substitute the value of h in the volume formula to get

$$V = x^2 h = x^2 \left(\frac{108 - x^2}{4x} \right) = 27x - \frac{x^3}{4}.$$

The first and second derivatives are

$$V' = 27 - \frac{3}{4}x^2$$

$$V'' = -\frac{3}{2}x$$

The critical points are

$$\begin{aligned} V' &= 0 \\ 27 - \frac{3}{4}x^2 &= 0 \\ \frac{3}{4}x^2 &= 27 \\ x^2 &= 36 \\ x &= \pm 6. \end{aligned}$$

Using the second derivative test, V' is negative at $x = 6$. So V is maximum at

$$x = 6.$$

The corresponding value of h is

$$h = \frac{108 - 36}{36} = 2.$$

Therefore the dimensions giving the maximum volume are $x = 6$ inches and $h = 2$ inches. □

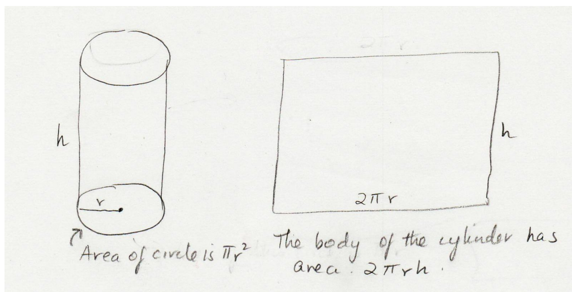
Example 11.3. (Maximizing Surface Area) A cylindrical can must be constructed to hold 1000 cubic centimeters of oil. Find the dimensions that minimize the cost of the materials needed to manufacture the can.

Solution: Suppose the radius of the can's base is r and height is h. Volume of the can is

$$V = \pi r^2 h.$$

The surface area of the can is the area of the top and bottom circles which is $2 \times \pi r^2$ and the area of the cylinderical part. This portion unfolds to a rectangle with one side h and the other side $2\pi r$, so its area is $2\pi r \times h$. Thus the surface area of a cylinderical can is

$$S = 2\pi r^2 + 2\pi r h.$$



The problem specifies that the volume is fixed at 1000 cubic centimeters. So

$$1000 = \pi r^2 h$$
$$h = \frac{1000}{\pi r^2}.$$

Substitute h is the surface area formula to get

$$S = 2\pi r(r + \frac{1000}{\pi r^2}) = 2\pi r^2 + \frac{2000}{r}.$$

The first and second derivatives are

$$S' = 4\pi r - \frac{2000}{r^2}$$

$$S'' = 4\pi + \frac{4000}{r^3}.$$

The critical points are obtained by setting

$$S' = 0$$
$$4\pi r - \frac{2000}{r^2} = 0$$
$$\frac{4\pi r^3 - 2000}{r^2} = 0$$

Setting the numerator zero gives

$$\begin{aligned} 4\pi r^3 - 2000 &= 0 \\ r^3 &= \frac{500}{\pi} \\ r &= \sqrt[3]{\frac{500}{\pi}} \\ &\approx 5.42. \end{aligned}$$

So the critical point is $r = 5.42$. [1] Using the Second Derivative Test, observe that A'' is positive at $r \approx 5.42$. So V is minimum at

$$r \approx 5.42.$$

The corresponding value of h is

$$h = \frac{1000}{\pi\sqrt[3]{5.42^2}} \approx 10.84.$$

Therefore, the dimensions of the can that minimize the surface area (and therefore the cost of materials) is $r \approx 5.42$ cm and $h \approx 10.84$ cm. [2]

EXAMPLE 11.4. A rectangular page must be constructed with 24 square inches of print, with top and bottom margins of 1.5 inches and left and right margins of 1 inch. What should the dimensions of the page be in order to use the least amount of paper?

Solution: Suppose the printed portion has length l and width w. The area of the printed portion is given as 24. So

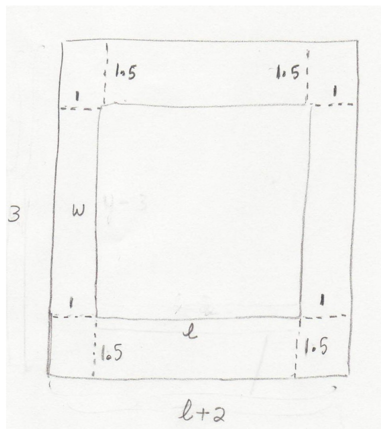
$$\begin{aligned} 24 &= lw \\ w &= \frac{24}{l}. \end{aligned}$$

[1]Note that $\frac{4\pi r^3 - 2000}{r^2} = 0$ is a rational equation which is zero when the numerator is zero. The numerator is the cubic equation $4\pi r^3 - 2000$. It has one real root and two complex roots. But r cannot be a complex number so we disregard the complex roots. Further, note that $\frac{4\pi r^3 - 2000}{r^2} = 0$ is undefined at $r = 0$ making it another critical point. But a solution with $r = 0$ would not make sense, so we can disregard it. These reasons must be known for a thorough understanding, even if forgetting them and finding only the real root gets you to the final answer.

[2] See http://samjshah.com/2012/05/31/a-calculus-optimization-poster-project/ for a Calculus Optimization project where students can implement this analysis for various grocery items such as soup cans, soda cans etc. to determine how "volume optimized" the cans are.

Since the left and right margins are 1 inch each, the length of the printed portion is $l-2$ and since the top and bottom margins are 1.5 inches each, the width is $w-2$. The area of the page is

$$A = (l+2)(w+3).$$



Substituting the value of w in the above formula gives

$$A = (l+2)(\frac{24}{l}+3) = 30 + 2l + \frac{72}{l}$$

The first and second derivatives are

$$A' = 2 - \frac{72}{l^2}$$

and

$$A'' = \frac{144}{x^3}.$$

The critical points are

$$\begin{aligned} 2 - \frac{72}{l^2} &= 0 \\ 2l^2 &= 72 \\ l^2 &= 36 \\ l &= \pm 6. \end{aligned}$$

Since a solution with $l = -6$ doesn't make sense, we may assume $l = 6$. Since A' is positive when $l = 6$, by the Second Derivative Test A' is minimized at

$$l = 6.$$

The corresponding w value is

$$w = \frac{24}{l} = 4.$$

Therefore, the dimensions of the printed area using the least paper are $l = 6$ inches and $w = 4$ inches. $\square$

When we look at the History of Mathematics we see that Mathematics advanced for practical reasons as well as through puzzles and games. The next set of optimization problems orginated as puzzles. All cultures came up with such problems more or less independently and devised ways of solving them using derivative ideas. For example, during the Edo period (1603 - 1867) in Japan, people lived in self-imposed isolation from all outside influences and during this time a tradition of writing short poems (haikus) and geometrical problems on wooden tablets and leaving them in temples as offerings emerged. This tradition was called **Sangaku**, which means mathematical tablet. For example, the famous Japanese poet Matsuo Basho (1644 - 1694) wrote the following haiku on a Sangaku tablet.

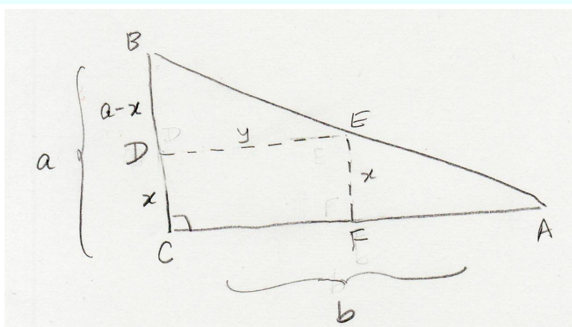
Year after year

On the monkey's face

A monkey's mask.

The next problem is a Sangaku optimization problem. Although the Japanese did not have the concept of derivative as we do, they managed to solve it using the same sort of ideas. [3]

Example 11.5. Find the area of the largest rectangle that can be embedded in a right triangle.



Solution. Let ABC be the right triangle with legs of length a and b as shown and let x and y be the length and width, respectively, of rectangle $CFED$ embedded in the triangle. Area of the rectangle is

$$A = xy$$

Observe that Triangle BDE is similar (angles are equal) to Triangle ABC, so the ratio of sides is the same. Therefore,

$$\frac{y}{b} = \frac{a-x}{a}.$$

So

$$y = \frac{b(a-x)}{a}.$$

[3] See http://www.cut-the-knot.org/pythagoras/Sangaku.shtml and http://www.cut-the-knot.org/pythagoras/QOptimizationSangaku.shtml. As the article notes, it is interesting that Japanese mathematicians of the Edo era wrote a lot about integration but little on the derivative, other than solving problems like this.

Substituting the value of y in the area formula gives

$$A = \frac{x}{y} = x\frac{b(a-x)}{a} = \frac{b}{a}(ax - x^2).$$

The first and second derivatives are

$$A' = \frac{b}{a}(a - 2x)$$

and

$$A'' = -2\frac{b}{a}$$

The critical point is

$$\begin{aligned} A' &= 0 \\ \frac{b}{a}(a-2x) &= 0 \\ a - 2x &= 0 \\ 2x &= a \\ x &= \frac{a}{2} \end{aligned}$$

Since A'' is always negative (a and b are lengths and therefore positive), by the Second Derivative Test, $x = \frac{a}{2}$ is a local maximum. So A is maximized at

$$x = \frac{a}{2}$$

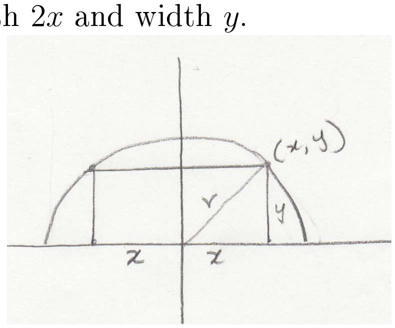
and the corresponding value of y is

$$y = \frac{b}{a}(a - \frac{a}{2}) = \frac{b}{2}.$$

Therefore, the area of the largest rectangle is $\frac{ab}{4}$.

Example 11.6. Find the area of the largest rectangle that can be inscribed in a semi-circle of radius r.

Solution: Draw a circle centered at $(0, 0)$ and radius r and inscribe a rectangle in it as shown. The rectangle has length $2x$ and width y.



The area of the rectangle is

$$A = 2xy.$$

The equation of the circle is

$$x^2 + y^2 = r^2.$$

Since the point (x, y) lies on the circle (as shown)

$$\begin{aligned} x^2 + y^2 &= r^2 \\ y &= \sqrt{r^2 - x^2} \end{aligned}$$

Substituting, the value of y in the area formula gives

$$A = 2xy = 2x\sqrt{r^2 - x^2}$$

The first and second derivatives are

$$\begin{aligned} A' &= 2\left[x\frac{1}{2\sqrt{r^2 - x^2}}(-2x) + \sqrt{r^2 - x^2}\right] \\ &= 2\left[\frac{-x^2}{\sqrt{r^2 - x^2}} - \sqrt{r^2 - x^2}\right] \\ &= 2\left[\frac{-x^2 + r^2 - x^2}{\sqrt{r^2 - x^2}}\right] \\ &= \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} \\ &= \frac{2r^2 - 4x^2}{\sqrt{r^2 - x^2}} \end{aligned}$$

and

$$\begin{aligned} A'' &= \frac{1}{r^2 - x^2}\left[\sqrt{r^2 - x^2}(-8x) - (2r^2 - 4x^2)\frac{-2x}{2\sqrt{r^2 - x^2}}\right] \\ &= \frac{1}{(r^2 - x^2)^{\frac{3}{2}}}\left[(r^2 - x^2)(-8x) - (2r^2 - 4x^2)(-x)\right] \\ &= \frac{1}{(r^2 - x^2)^{\frac{3}{2}}}\left[-8xr^2 + 8x^3 + 2xr^2 - 4x^3\right] \\ &= \frac{1}{(r^2 - x^2)^{\frac{3}{2}}}\left[-6xr^2 + 4x^3\right] \end{aligned}$$

The critical points are the points where A' is undefined or 0. Observe that A' is undefined when $x = \pm r$, but it makes no sense geometrically, so we disregard it. Setting the numerator equal to zero gives

$$\begin{aligned} 2(r^2 - 2x^2) &= 0 \\ x &= \pm\frac{r}{\sqrt{2}} \end{aligned}$$

Observe that when $x = \frac{r}{\sqrt{2}}$ the value of A'

$$\begin{aligned} A' &= \frac{1}{(r^2 - \frac{r^2}{2})^{\frac{3}{2}}}\left[-6\frac{r}{\sqrt{2}}r^2 + 4\frac{r^3}{8}\right] \\ &= \frac{1}{r^3}\left[-\frac{6}{\sqrt{2}}r^3 + \frac{r^3}{2}\right] \\ &= \frac{1}{r^3\sqrt{2}}(-11r^3) \end{aligned}$$

Since A' is negative when $\frac{r}{\sqrt{2}}$, by the Second Derivative Test, it is a local maximum. So A is maximized when

$$x = \frac{r}{\sqrt{2}}$$

and the corresponding y value is

$$y = \sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2} = \frac{r}{\sqrt{2}}.$$

The area of the largest rectangle is

$$A = \frac{2r}{\sqrt{2}}\frac{r}{\sqrt{2}} = r^2$$

□

The differentiation in the above problem would have been easier if we were given a specific value of r. In the next two problems we reduce the generality of the question by talking about specific geometrical objects and specific points.

EXAMPLE 11.7. Find the point on the line $y = 4x + 7$ that is closest to the origin $(0, 0)$.

Solution: Let (x, y) be a point on the line $y = 4x + 7$ and let d be the distance from (x, y) to the origin $(0, 0)$.

$$d = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}.$$

It would be difficult to maximize d, so we maximize d^2. (This portion is not obvious and is explained in further detail after the solution.) Substitute the value of $y = 4x + 7$ to get

$$d^2 = x^2 + (4x + 7)^2 = x^2 + 16x^2 + 56x + 49 = 17x^2 + 56x + 49$$

For the purpose of applying the Second Derivative Test let us call this function

$$f(x) = 17x^2 + 56x + 49$$

$$f'(x) = 34x + 56$$

The critical point is $x = -\frac{56}{34} \approx -1.65$. Since the second derivative is $f''(x) = 34$, which is positive, $x = -1.64$ is a minimum. The corresponding y is $y = 4x + 7 = 0.41$. Therefore the point on the graph closest to the origin is $(-1.65, 0.4)$. □

Now let us understand in more detail why we can naximize d^2 in place of d in the above problem. Intuitively, consider an increasing function f and let $g = f^2$. Then g is also an increasing function. So f and g will have the same local extrema. Formally, a function f is **strictly increasing** on an interval $[a, b]$ if, for any $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$, $f(x_1) < f(x_2)$.

Theorem 11.8. Suppose f is a differentiable function on an interval $[a, b]$ and that g is a differentiable, strictly increasing function on the interval $[f(a), f(b)]$. Let $h = g \circ f$, that is $h(x) = g(f(x))$. Then the local maxima and minima of h are the same as the local maxima and minima of f, respectively.

Proof. Suppose c is a local maximum of f. By definition of local maximum for some interval (d, e) containing c,

$$f(x) < f(c)$$

for all $x \in (d, e)$ distinct from c. Since g is strictly increasing,

$$g(f(x)) < g(f(c)).$$

Therefore c is a local maximum for $g \circ f$.

Similarly, if c is a local minimum of f, then c must also be a local minimum for h. □

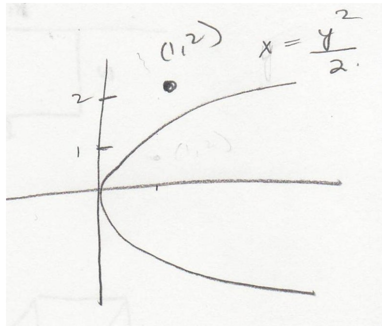
EXAMPLE 11.9. Find the point on the parabola $x = \frac{y^2}{2}$ that is closest to the point $(1, 4)$.

Solution: Let (x, y) be a point on the parabola $x = \frac{y^2}{2}$ and let f be the square of the distance of (x, y) from $x = \frac{y^2}{2}$. Observe that

$$f = (x - 1)^2 + (y - 4)^2$$

and replacing $x = \frac{y^2}{2}$ gives

$$f = (\frac{1}{2}y^2 - 1)^2 + (y - 4)^2.$$



The first and second derivatives are

$$f' = 2(\frac{1}{2}y^2 - 1)y + 2(y - 4) = y^3 - 2y + 2y - 8 = y^3 - 8$$

$$f'' = 3y^2$$

The real root of $y^3 - 8 = 0$ is $y = 2$. So f has a critical point at $y = 2$. Since f'' is positive when $y = 2$, by the Second Derivative Test, $y = 2$ is a local minimum. So f is minimized by $y = 2$ and the corresponding value of x is $x = \frac{y^2}{2} = \frac{2^2}{2} = 2$. Therefore the point on the parabola closest to $(1, 2)$ is $(2, 2)$.

Example 11.10. Find the points on the ellipse $4x^2 + y^2 = 4$ farthest from $(1, 0)$.

Solution: Let (x, y) be a point on the ellipse and let f be the square of the distance from (x, y) to $(1, 0)$. Then

$$f = (x - 1)^2 + y^2.$$

Using the equation of the ellipse, we can write $y^2 = 4 - 4x^2$, so we can substitute in f to obtain

$$f = (x - 1)^2 + 4 - 4x^2 = 5 - 2x - 3x^2$$

The first and second derivatives are

$$f'' = -2 - 6x$$

$$f''(x) = -6.$$

The critical point is $x = -\frac{1}{3}$. Since f'' is negative, $x = -\frac{1}{3}$ is a local maximum. The corresponding y values are

$$\pm\sqrt{4 - 4\left(\frac{1}{9}\right)} = \pm\sqrt{\frac{32}{9}} = \pm\frac{4\sqrt{2}}{3}.$$

Therefore, the farthest points from $(1, 0)$ on the ellipse are $(-\frac{1}{3}, \pm\frac{4\sqrt{2}}{3})$. □

Practice Problems

(1) Find two positive numbers that satisfy the following constraints:

a) The sum is 100 with maximum product

b) the product is 192 with maximum sum

c) The product is 192 and the sum of the first plus three times the second is a minimum

d) The sum of the first and twice the second is 100 with maximum product

(2) A farmer as 2400 meters of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

(3) A farmer wants to fence a rectangular pasture adjacent to a river. The pasture must contain 180,000 square meters in order to provide enough grass for the herd. What dimensions would require the least amount of fencing if no fencing is needed along the river?

(4) A manufacturer wants to design an open box having a square base and surface area of 108 square inches. What dimensions will produce a box with maximum volume?

(5) Find the dimensions of a rectangular solid with a square base of maximum volume if the surface area is 160 square inches.

(6) A rectangular page has 24 square inches of print. Margins at the top and bottom are 1.5 inches and on the left and right are 1 inch What should the dimensions of the page be so that the smallest number of pages are used?

(7) A rectangular page is to contain 30 square inches of print. The margins on each side are 1 inch. Find the dimensions of the page so that the smallest amount of paper is used.

(8) A cylindrical can is to be made to hold 1000 cubic centimeters of oil. Find the dimensions that minimize the cost of the metal to manufacture the can.

(9) Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

(10) Find the area of the largest rectangle that can be inscribed in a semi-circle of radius r.

(11) A rectangular box with a square base and no top is to have a volume of 108 cubic inches. Find the dimensions of the box that require the smallest amount of material.